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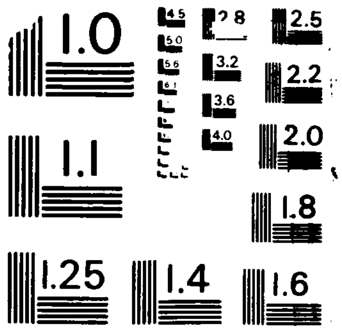

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COORDINATED SCIENCE LABORATORY
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Decision and Control Laboratory

REPRESENTATION, ANALYSIS, AND DESIGN OF MULTIRATE DISCRETE-TIME CONTROL SYSTEMS

Kevin Lee Buescher

UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

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Multirate discrete-time systems, systems in which sampling and discrete-time calculations are performed at two or more rates, arise in a variety of applications. The multirate character of the system may be intrinsic, due to digital subsystems operating at multiple rates. Alternately, the multirate nature of the system may be induced by the addition of sensors, actuators, and discrete-time control structures at different rates for the purposes of economy or performance.

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REPRESENTATION, ANALYSIS, AND DESIGN OF MULTIRATE DISCRETE-TIME CONTROL SYSTEMS

BY

KEVIN LEE BUESCHER

B.S., University of Illinois, 1986

THESIS

Submitted in partial fulfillment of the requirements
for the degree of Master of Science in Electrical Engineering
in the Graduate College of the
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SYMBOL TABLE

Symbol	Meaning
$\mathbb{N}$	the natural numbers: {1, 2, 3,... }
$\mathbb{Q}$	the rational numbers
$\mathbb{R}^{n \times m}$	an $n \times m$ matrix with real-valued entries
$\mathbb{Z}^+$	the nonnegative integers: {0, 1, 2,... }
$\forall$	for all
n mod p	n modulo p: $p \times$ (fractional portion of n/p)
SISO	Single-Input–Single-Output
MIMO	Multiple-Input–Multiple-Output
BIBO	Bounded-Input–Bounded-Output
GCD	Greatest Common Divisor
LCM	Least Common Multiple
LCP	Least Common Period
T	the LCP of a multirate system
τ	the fundamental period of a multirate system
P	T/τ
A'	transpose of the matrix A
A_{ij}	i,j th element of the matrix A
I_n	the $n \times n$ identity matrix
0_m	an $m \times m$ matrix of zeroes
$0_{m \times n}$	an $m \times n$ matrix of zeroes
♦ ♦ ♦	end of theorem, procedure, example, etc.

CHAPTER 1

INTRODUCTION

Multirate discrete-time systems, systems in which sampling and discrete-time calculations are performed at two or more rates, arise in a variety of applications. The multirate character of the system may be intrinsic, due to digital subsystems operating at multiple rates. Alternately, the multirate nature of the system may be induced by the addition of sensors, actuators, and discrete-time control structures at different rates for the purposes of economy or performance. Two practical examples of designs resulting in a multirate system are an idle speed control for an internal combustion engine (Powell et al., 1987) and a quadruplex videotape recorder (Rao, 1979).

Previous studies of multirate systems began with the study of multirate sampled-data systems by Kranc (1957), which employed transfer function techniques later improved by Coffey and Williams (1966) and Boykin and Frazier (1975). These works presented complicated methods which served the purpose of assessing the stability of a system composed of continuous-time subsystems and samplers at multiple rates. Kalman and Bertram (1959) used state space methods to study sampling systems of great generality. Although Kalman and Bertram demonstrated that a state space representation exists for almost any hybrid system incorporating a variety of sampling schemes, the generality of the systems involved precluded the presentation of systematic and efficient methods of obtaining a state space representation for a given, fixed system. Meyer and Burrus (1975) introduced the concept of block processing to perform time and frequency domain analyses on individual single-rate, periodic time-varying and multirate (inputs at one rate, outputs at another) digital filters. Recently, Araki and Yamamoto (1986) applied a mutation of block processing to the analysis of a continuous-time system with outputs sampled at multiple rates and fed back through a constant gain matrix to the inputs, which were sampled and held at multiple rates. Currently, efficient techniques of analysis and design algorithms for multirate systems composed of both continuous-time and discrete-time subsystems are not available.

Inspired by Kalman and Bertram (1959) and beginning with methods akin to the block processing used by Meyer and Burrus (1975), this work presents a systematic, efficient method of obtaining a "time-invariant" representation, referred to as the **T-expanded representation**, of a class of composite multirate systems containing both continuous-time and discrete-time subsystems. In recent years, results have appeared in the literature pertaining to periodic discrete-

time systems, with multirate systems cited as an (unqualified) example of such a system. In response, this work proposes a procedure for obtaining a periodic discrete-time representation, termed the **M-varying representation**, of a multirate system. Analysis of multirate systems via the T-expanded representation by time-invariant techniques is shown to reveal important characteristics of a certain M-varying representation of the multirate system. Examples of the design of controllers for multirate systems by conventional analytical techniques, with minor modifications, are presented. The variable component method is applied to multirate systems to provide an iterative means of building composite controllers.

Chapter 2 develops a notation suited to T-expanded representations and details procedures useful in obtaining a T-expanded representation. The notation introduced in Chapter 2 is used throughout the remaining chapters and provides a means of discussing a variety of concepts related to multirate systems in a concise manner. In addition to describing the behavior of a multirate system over lengths of time other than its period, the periodic representations developed in Chapter 3 provide a theoretical tool for deducing the interperiod behavior of a multirate system from its T-expanded representation. Chapter 4 discusses the stability, controllability, reconstructibility, stabilizability, and detectability of multirate systems and emphasizes the properties which a periodic representation of a multirate system inherits from the T-expanded representation of that system. Chapter 5 applies the developments of Chapters 2, 3, and 4 to the analytical design of multirate controllers. A time-invariant technique, the variable component method, is applied to multirate systems in Chapter 6 via the T-expanded representation. The material in Chapter 5 and Chapter 6 serves as an example of the manner in which results for standard discrete-time systems can be extended to multirate systems.

CHAPTER 2

TIME-INVARIANT REPRESENTATION OF MULTIRATE SYSTEMS

This chapter presents a systematic method for obtaining a single time-invariant state space or transfer function representation, called a T-expanded representation, for members of a class of composite systems which employ sampling at multiple rates. This method first converts the block diagram of a multirate system into a structurally similar block diagram of time-invariant discrete-time subsystems through state space calculations easily performed using any software package capable of discretization of continuous-time systems at a single rate and matrix multiplication, addition, and composition. The reduction of this single-rate block diagram to a state variable description or transfer function matrix is then a standard problem.

2.1 Systems Admissible for Analysis

The methods to be developed will apply to a general class of systems with multiple sampling rates. To simplify the exposition, the multirate systems considered, unless otherwise noted, will be as follows:

Assumption 2.1:

a. The ratio of any two sampling periods in the system is a rational number.

b. All samplers are in synchronism at $t = 0$ and are followed by zero-order holds of the same period

c. Discrete-time subsystems are envisioned as sampling their inputs and producing zero-order hold-type outputs (in synchronism with the samplers in part (b) above at $t = 0$.)

d. The outputs of zero-order holds as functions of time are continuous from the right at their associated sampling instants.

e. The entire system can be partitioned into subsystems which are either discrete-time or continuous-time subsystems with only samplers, summers, and branch points (a point on the block diagram where a line branches into two or more lines) connecting these subsystems together on the block diagram. In addition, if

there is a path on the block diagram between any output of one of these subsystems and any input to one of these subsystems, then this path must satisfy one or more of the following: the path originates at a discrete-time subsystem; the path terminates at a discrete-time subsystem; the path passes through one or more samplers.

f. There is no path from (to) any external input (output) of the entire multirate system to (from) any of the continuous-time subsystems in part (e) above which does not pass through a sampler. In addition, there is no direct path from an external input to an external output of the multirate system which does not pass through a sampler. These properties will be loosely stated in the following as "the external inputs and outputs of the multirate system are sampled."

g. The subsystems in part (e) above are linear, time invariant, and causal, and may be MIMO.

♦♦♦

Assumption 2.1(a) is the necessary and sufficient condition for a multirate system composed of time-invariant subsystems to have a finite period. Satisfaction of Assumption 2.1(e) is possible for essentially any connection of continuous-time and discrete-time elements and may entail minor block diagram manipulations, to be detailed later in this section. If a system does not satisfy part (f), the introduction of samplers into the block diagram by modelling judgments may serve to approximate the actual system adequately and satisfy Assumption 2.1(f). Assumption 2.1(g) will eventually be relaxed to include time-varying periodic discrete-time subsystems. In summary, the essential assumptions to keep in mind are that all external signals (inputs/outputs) are sampled and held and that the ratio of any two sampling rates in the system is a rational number.

Although the term "linear multirate discrete composite system" more aptly describes a system satisfying Assumption 2.1, **multirate system** will denote such a system hereafter.

Example 2.1:

The multirate system shown in Figure 2.1 appears to violate Assumption 2.1(e) due to the direct link between the two continuous-time subsystems σ_1 and σ_2; $y_1(s)$ passes through a summer and a branch point, but not through a sampler, before reaching $u_3(s)$. Let σ_1 and σ_2 be represented by the transfer function relationships

$$\begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} N_1(s)/D_1(s) \\ N_2(s)/D_1(s) \end{bmatrix} u_1(s) \quad \text{and} \quad y_4(s) = [N_3(s)/D_2(s) \quad N_4(s)/D_2(s)] \begin{bmatrix} u_3(s) \\ u_4(s) \end{bmatrix}.$$

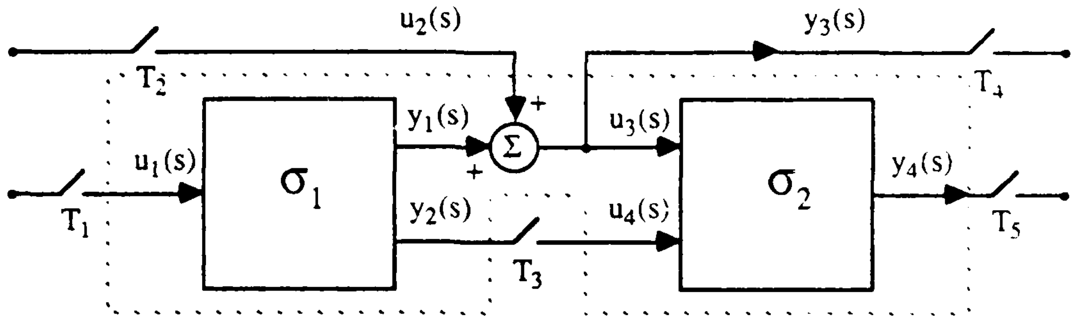

Figure 2.1. A multirate system.

Using $u_3(s) = y_1(s) + u_2(s)$ and $y_3(s) = u_3(s)$, it follows that

$$\begin{bmatrix} y_2(s) \\ y_3(s) \\ y_4(s) \end{bmatrix} = \begin{bmatrix} N_2(s)/D_1(s) & 0 & 0 \\ N_1(s)/D_1(s) & 1 & 0 \\ N_3(s)N_1(s)/D_2(s)D_1(s) & N_3(s)/D_2(s) & N_4(s)/D_2(s) \end{bmatrix} \begin{bmatrix} u_1(s) \\ u_2(s) \\ u_4(s) \end{bmatrix}.$$

Thus, the multirate system shown in Figure 2.1 may be partitioned by considering the area inside the dashed line in Figure 2.1 as a single continuous-time subsystem, σ_3, as shown in Figure 2.2. The multirate system in Figure 2.2 satisfies Assumption 2.1(e).

♦♦♦

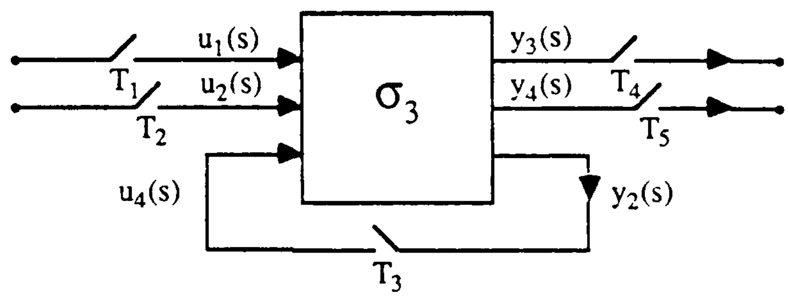

Figure 2.2. The system in Figure 2.1 redrawn.

Example 2.1 provides a clue to a technique for partitioning a multirate system so that Assumption 2.1(e) is satisfied. If an initial partitioning of the system into continuous-time and discrete-time subsystems (connected by only summers, samplers, and branch points) does not satisfy Assumption 2.1(e), then by elimination a path must be connecting a continuous-time subsystem, σ_1, to a continuous-time subsystem, σ_2, which may pass through summers and branch points, but does not pass through any samplers (σ_1 may be the same subsystem as σ_2.) Define a new continuous-time subsystem, σ_3, consisting of the subsystems σ_1 and σ_2 and their individual

inputs and outputs (u_1,u_4,y_2, and y_4 in Example 2.1,) except for the offending pair in question (y_1 and u_3 in Example 2.1.) The subsystem σ_3 must also be provided with additional inputs (u_2 in Example 2.1) representing other inputs to any summers the path passes through and additional outputs (y_3 in Example 2.1) representing other outputs of any branch points the path passes through. This process can be repeated until the entire multirate system satisfies Assumption 2.1(e).

2.2 The Least Common Period of a Multirate System

Given a set of N nonzero sampling periods $\{T_1,...,T_N\}$, assume that the ratio of any two of these sampling periods is a rational number. Consider the ratio of each sampling period to a particular sampling period, say T_1:

$$\frac{T_1}{T_i} = \frac{r_{1i}}{q_{1i}}, \quad i = 1,...,N,$$

for some r_{1i}, $q_{1i} \in \mathbb{N}$, where the Greatest Common Divisor (see Niven, 1980) of r_{1i} and q_{1i} is 1 for each i. Let R_1 be the Least Common Multiple of the set $\{r_{1i} : i = 1,...,N\}$, $\tau_1 = T_1/R_1$, and

$$p_{1i} = \frac{q_{1i}R_1}{r_{1i}}, \; i = 1,...,N.$$

Note that $p_{1i} \in \mathbb{N}$ because $R_1/r_{1i} \in \mathbb{N}$. Then

$$T_i = \frac{T_1 q_{1i}}{r_{1i}} = \left(\frac{T_1}{R_1}\right) q_{1i} \left(\frac{R_1}{r_{1i}}\right) = p_{1i}\tau_1 \;, \; i = 1,...,N.$$

The process of obtaining τ_1 and the p_{1i}'s will be referred to as **normalization** with respect to T_1.

Theorem 2.1:

If the sampling periods are normalized with respect to T_k (giving $T_i = p_{ki}\tau_k$, $i = 1,...,N$) and again with respect to T_j (giving $T_i = p_{ji}\tau_j$, $i = 1,...,N$,) then $\tau_k = \tau_j$ and $p_{ki} = p_{ji}$, $i = 1,...,N$. (See proof in the Appendix.)

♦ ♦ ♦

Thus, the sampling periods can be uniquely represented as $T_i = p_i\tau$, $i = 1,...,N$, regardless of the particular sampling period used in the normalization process. Each sampling period in the system is a multiple of τ; τ will be referred to as the **fundamental period** of the multirate system.

Since the subsystems in the multirate system are time invariant, the periodicity of the entire system will be established if it can be shown that the samplers, which are in synchronism at $t = 0$, are again in synchronism at some time $t = T$. For a system with sampling periods $\{T_1,...,T_N\}$, let $P = \mathrm{LCM}\,\{p_i : i = 1,...,N\}$. At times $t = mP\tau$, $m \in \mathbb{Z}^+$, all samplers in the system will sample:

$$\frac{mP\tau}{T_i} = \frac{mP}{p_i} \in \mathbb{Z}^+ \text{ for } i = 1,...,N.$$

Thus, the multirate system is periodic with period $T = P\tau$; in fact, this is the Least Common Period (LCP) of the multirate system.

Theorem 2.2:

$T = \tau\,(\mathrm{LCM}\ \{p_i : i = 1,\ldots,N\}) = P\tau$ is the shortest length of time over which a multirate system with sampling periods $\{T_1,\ldots,T_N\}$ is periodic. (See proof in the Appendix.)

♦♦♦

In the remainder of this work, the symbols T, P, τ, and p_i will be implicitly associated with the meanings assigned to them above. A procedure is now given for normalizing a set of sampling periods $\{T_1,\ldots,T_N\}$ as $T_i = p_i\tau$ and finding their LCP.

Procedure 2.1:

a. Arbitrarily choose a sampling period T_k from the set and express the sampling periods as $T_i = T_k q_{ki}/r_{ki}$, $i = 1,\ldots,N$, where $q_{ki}, r_{ki} \in \mathbb{N}$ and $\mathrm{GCD}\ \{q_{ki}, r_{ki}\} = 1$ for each i.

b. Set $R_k = \mathrm{LCM}\ \{r_{ki} : i = 1,\ldots,N\}$

c. Set $\tau = T_k/R_k$ and $p_i = q_{ki}(R_k/r_{ki})$, $i = 1,\ldots,N$.

d. Set $P = \mathrm{LCM}\ \{p_i : i = 1,\ldots,N\}$ and $T = P\tau$.

♦♦♦

The following result will be the key, in Section 2.8, for establishing a certain modularity property of multirate systems.

Theorem 2.3:

Let the LCP associated with $\{T_1,\ldots,T_k\}$ be T(k) and that associated with $\{T_1,\ldots,T_k,T_{k+1}\}$ be T(k+1). Then $T(k+1)/T(k) \in \mathbb{N}$. (See proof in the Appendix.)

♦♦♦

In words, the effect of adding a new sampling period to a multirate system is to increase the LCP by an integer multiple.

Example 2.2:

Suppose N = 4 and the sampling periods are: $T_1 = 3/5$ sec, $T_2 = 1/7$ sec, $T_3 = 1/3$ sec, and $T_4 = 300$ msec. If the periods are normalized with respect to T_1, Procedure 2.1 yields

$$T_1 = (1/1)T_1,\ T_2 = (5/21)T_1,\ T_3 = (5/9)T_1,\ T_4 = (1/2)T_1$$

$$R_1 = \mathrm{LCM}\ \{1,21,9,2\} = 7 \times 3 \times 3 \times 2 = 126$$

$$\tau = T_1/R_1 = 1/210 \text{ sec}$$

$$p_1 = 126(1/1) = 126,\ p_2 = 126(5/21) = 30,\ p_3 = 126(5/9) = 70,\ p_4 = 126(1/2) = 63$$

$$P = \mathrm{LCM}\{126,30,70,63\} = 126 \times 5 \times 7 = 4410$$

$$T = 4410(1/210) = 21 \text{ sec}$$

The reader may wish to verify that $p_1,\ldots,p_4,\tau,P$, and T are invariant under normalization with respect to T_2,T_3, or T_4.

♦♦♦

2.3 Expansion of Discrete-Time Signals and Systems

The members of the class of multirate systems under consideration are time varying, but periodic with period T. Knowledge of the state of such a multirate system at time $t = 0$ and the inputs to the system over the interval of time $[0,T)$ is sufficient to determine its state at time $t = T$ and its outputs over the interval $[0,T)$. By the periodicity of such a multirate system, the manner in which its state at time $t = nT$ and its output over $[(n-1)T,nT)$ are determined from its state at time $t = (n-1)T$ and input over $[(n-1)T,nT)$ is identical for each $n \in \mathbb{N}$. Thus, the behavior of such a multirate system over all time is time invariant in terms of the description of the system's behavior over one period. The price paid for this time-invariant description is that all input values to the system and all output values from the system during one period must be accounted for. This entails expanding a single input line into many fictitious inputs which represent its values over the period.

The result of collecting all values of a discrete-time signal over amounts of time of length L will be referred to as the **L-expanded version** of that signal. The signals in question may be vector signals with individual components at different rates. To prevent the number of components in the L-expanded version of a signal from changing with time, L must be an integer multiple of the period of each component of the signal. Discrete-time signals will be denoted by lower-case letters and the same letter capitalized will represent their expanded versions.

At this point it is convenient to introduce the concept of a signal bundle. A **signal bundle** is a vector of discrete-time signals at the same rate which may be a portion of a larger vector of signals but is distinguished in some manner from the rest of the vector. The block diagram of the system primarily determines the grouping of signals into bundles. Before applying the methods presented here, certain manipulations will be performed on the block diagram of the system. The lines drawn as inputs or outputs of subsystems in this modified block diagram will each be designated as a bundle of signals; each line may actually represent an entire vector of signals. Later developments will reveal the utility of using bundles of signals and properly explain which signals to place in bundles.

Let $L/T_1 = l_1 \in \mathbb{N}$. The L-expanded version of a bundle of signals $u(nT_1)$, $n = 0,1,\ldots$ is the vector of l_1 blocks $U(kL)$:

$$U(kL) = \begin{bmatrix} u(kl_1T_1) \\ u((kl_1+1)T_1) \\ \vdots \\ u((kl_1+l_1-1)T_1) \end{bmatrix}, \; k = 0,1,\ldots .$$

$U(kL)$ will be thought of as a bundle of signals with rate $1/L$. The set $\{U(kL) : k = 0,1,\ldots\}$ contains all of the values in $\{u(nT_1) : n = 0,1,\ldots\}$, so the expanded version retains all the information in the original signal. Capital letters will denote expanded versions of signals hereafter, and the capitalized portion of an expanded signal's argument (L in this case, but T and M later on) will be a real number which is an integer multiple of τ and denotes the interval of time over which expansion has been performed. As a slight abuse of notation, let $u(kL) = u(kl_1T_1)$. Then the L-expanded version of $u(nT_1)$ can be written more conveniently as

$$U(kL) = \begin{bmatrix} u(kL) \\ u(kL+T_1) \\ \vdots \\ u(kL+L-T_1) \end{bmatrix}.$$

The expanded version of a signal may be expanded again to yield an expanded signal. The procedure for expanding an expanded signal parallels that for expanding a normal signal, with expanded bundles treated as bundles. As an example, if $M/L = m \in \mathbb{N}$, then the M-expanded version of $U(kL)$ coincides with the M-expanded version of $u(nT_1)$ and is given by

$$U(jM) = \begin{bmatrix} U(jmL) \\ U((jm+1)L) \\ \vdots \\ U((jm+m-1)L) \end{bmatrix} = \begin{bmatrix} u(jml_1T_1) \\ u((jml_1+1)T_1) \\ \vdots \\ u((jml_1+ml_1-1)T_1) \end{bmatrix}, \; j = 0,1,\ldots .$$

To see the equivalence, note that the first component of $U(jmL)$ is $u((jm)l_1T_1) = u(jml_1T_1)$ and the last component of $U((jm+m-1)L)$ is $u(((jm+m-1)l_1+l_1-1)T_1) = u((jml_1+ml_1-1)T_1)$. The equivalence of the M-expanded version of a signal and the M-expanded version of the L-expanded version of that signal obviates the need for additional notation to distinguish between the two.

Let y be a vector of signals

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix},$$

where each y_i is a bundle of signals with period T_i. If L is an integer multiple of each T_i (for

example, $L = LCP\ \{T_1,...,T_q\}$,) then the L-expanded version of y is

$$Y(kL) = \begin{bmatrix} Y_1(kL) \\ Y_2(kL) \\ \vdots \\ Y_q(kL) \end{bmatrix}, \quad k = 0,1,\ldots\ ,$$

where each $Y_i(kL)$ is the L-expanded version of y_i. Although Y(kL) consists of many of the bundles y_i, Y(kL) will be thought of as being composed of the q bundles $Y_i(kL)$. Note that each bundle of signals has been expanded in place. Thus, the boundaries between bundles have been maintained.

Repeated iteration of the equations governing a simple discrete-time system reveals the behavior of such a system over units of time of length L and forms a basis for determining the behavior of more complex systems over units of time of length L. Consider the single-rate subsystem with realization (A,B,C,D) and period T_i, where $u(nT_i)$ and $y(nT_i)$ are each a single bundle of signals:

$$\begin{aligned} x((n+1)T_i) &= A\,x(nT_i) + B\,u(nT_i) \\ y(nT_i) &= C\,x(nT_i) + D\,u(nT_i) \end{aligned} \quad , n = 0,1,\ldots\ . \tag{2.1}$$

Since iteration of a discrete-time state equation can only be performed an integer number of times, let L be chosen so that $L/T_i = l_i \in \mathbb{N}$. Note that $x(kL+l_iT_i) = x((k+1)L)$. Expressions for $x((k+1)L)$ and $Y(kL)$ in terms of $x(kL)$ and $U(kL)$ are desired. By direct computation,

$$x(kL+T_i) = A\,x(kL) + B\,u(kL)$$
$$y(kL) = C\,x(kL) + D\,u(kL)$$

$$\begin{aligned} x(kL+2T_i) &= A\,x(kL+T_i) + B\,u(kL+T_i) \\ &= A^2\,x(kL) + AB\,u(kL) + B\,u(kL+T_i) \end{aligned}$$

$$\begin{aligned} y(kL+T_i) &= C\,x(kL+T_i) + D\,u(kL+T_i) \\ &= CA\,x(kL) + CB\,u(kL) + D\,u(kL+T_i) \end{aligned}$$

$$\vdots$$

$$x((k+1)L) = x(kL+l_iT_i) = A^{l_i}\,x(kL) + \sum_{m=1}^{l_i} [A^{(l_i-m)}B\,u(kL+(m-1)T_i)]$$

$$y(kL+(l_i-1)T_i) = CA^{(l_i-1)}\,x(kL) + D\,u(kL+(l_i-1)T_i) + \sum_{m=1}^{l_i-1} CA^{(l_i-m-1)}B\,u(kL+(m-1)T_i)\ .$$

After placing these equations in matrix form and using the L-expanded versions of u and y,

$$x((k+1)L) = A^{l_i}\, x(kL) + \left[\; A^{(l_i-1)}B \quad A^{(l_i-2)}B \quad \ldots \quad AB \quad B \;\right] U(kL)$$

$$Y(kL) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{(l_i-1)} \end{bmatrix} x(kL) + \begin{bmatrix} D & 0 & 0 & \ldots & 0 \\ CB & D & 0 & \ldots & 0 \\ CAB & CB & D & \ldots & 0 \\ \vdots & \vdots & . & & \vdots \\ CA^{(l_i-2)}B & \ldots & \ldots & CB & D \end{bmatrix} U(kL)\,. \tag{2.2}$$

These equations have the form of a time-invariant discrete-time system with period L:

$$\begin{array}{c} x((k+1)L) = A_e\, x(kL) + B_e\, U(kL) \\ Y(kL) = C_e\, x(kL) + D_e\, U(kL) \end{array}, \; k = 0,1,\ldots\,.$$

The matrices (A_e,B_e,C_e,D_e) will be referred to as the **single-bundle L-expanded representation** of (2.1), where "single-bundle" refers to the single-bundle nature of the inputs and outputs. Since in this form the system operates on blocks of input values and produces blocks of output values, the concepts employed above are often called block processing in the literature; see Meyer and Burrus (1975) and Burrus (1972).

The L-expanded representation of a single-rate discrete-time system with inputs and outputs that are vectors of bundles may be derived using manipulations performed in finding single-bundle L-expanded representations. Later sections will show that multirate discrete-time systems and multirate sampled continuous-time systems can be expressed as single-rate discrete-time systems, so the qualifier "single-rate" above does not restrict the applicability of the results in this section.

Let u and y be vectors

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \text{ and } y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix},$$

where each u_j, $j = 1,\ldots,m$ and each y_i, $i = 1,\ldots,q$ are bundles of signals, and all bundles are at the same rate, $1/T_j$. Consider a single-rate discrete-time system at rate $1/T_j$:

$$\begin{array}{c} x((m+1)T_j) = A\, x(mT_j) + B\, u(mT_j) \\ y(mT_j) = C\, x(mT_j) + D\, u(mT_j) \end{array}, \; m = 0,1,\ldots\,. \tag{2.3}$$

Let $L/T_j \in \mathbb{N}$ and B, C, and D be partitioned to conform with the bundles in u and y:

$$B = [\; b_1 \quad b_2 \quad \ldots \quad b_m \;],\; C = \begin{bmatrix} c_1 \\ \vdots \\ c_q \end{bmatrix},\text{ and } D = \begin{bmatrix} d_{11} & \ldots & d_{1m} \\ \vdots & & \vdots \\ d_{q1} & \ldots & d_{qm} \end{bmatrix}.$$

For each $j \in \{1,\ldots,m\}$ and $i \in \{1,\ldots,q\}$, let $(A_e,b_{je},c_{ie},d_{ije})$ be the single-bundle L-expanded representation of (A,b_j,c_i,d_{ij}). By superposition,

$$x((k+1)L) = A_e\, x(kL) + B_e\, U(kL)$$
$$Y(kL) = C_e\, x(kL) + D_e\, U(kL),$$

where

$$B_e = [\ b_{1e}\ \ b_{2e}\ \ \cdots\ \ b_{me}\ ],\ C_e = \begin{bmatrix} c_{1e} \\ \vdots \\ c_{qe} \end{bmatrix},\ \text{and } D_e = \begin{bmatrix} d_{11e} & \cdots & d_{1me} \\ \vdots & & \vdots \\ d_{q1e} & \cdots & d_{qme} \end{bmatrix},$$

and $Y(kL)$, $U(kL)$ are the L-expanded versions of y and u. In other words, (A_e,B_e,C_e,D_e) is the **L-expanded representation** of (2.3). B_e, C_e, and D_e are partitioned conformal with the expanded bundles composing $U(kL)$ and $Y(kL)$. As in the case of signals, L-expanded representations may be expanded again by thinking of (A_e,B_e,C_e,D_e) as a realization for a single-rate discrete-time system with rate 1/L.

For a system where the inputs and outputs have multiple bundles, finding (A_e,B_e,C_e,D_e) by merely using the same formulas as for the single-bundle expansion would be much easier, but this results in values over time L of each signal bundle being widely scattered throughout $U(kL)$ and $Y(kL)$. By performing the expansion as shown, values over time L of each particular signal bundle are adjacent in $U(kL)$ and $Y(kL)$. This greatly facilitates the connection of expanded systems. The partition of $U(kL)$ and $Y(kL)$ into expanded bundles in this form of system expansion reflects the designation of bundles as input and output lines on a block diagram.

The expanded representation possesses many of the characteristics of the original realization. It is easily verified that the number of states, stability, reachability, and observability of the original realization are all preserved by expansion. A feature of expanded representations that complicates certain control applications is that for a system with no direct feedthrough, $(A,B,C,0)$, the expanded representation will have D_e nonzero in general. Computationally, expansion involves only multiplication, composition, and storage of matrices. For large l_i, storing the distended matrices of the expanded representation may present difficulties: B_e and C_e each grow linearly with l_i, and the size of D_e increases as the square of l_i.

The expanded representation of a subsystem represents a step toward the goal of finding a time-invariant description of a multirate system. To complete the process of finding a time-invariant description of a multirate system, the expanded representation of multirate sampled continuous-time systems will be found, and methods for combining expanded representations will be developed.

2.4 Expanding Multirate Sampled Continuous-Time Systems

To find the expanded representation of a multirate sampled continuous-time system, it must be discretized first. To perform discretization, each input or output of the system must be sampled at only one rate. Inserting additional samplers that do not alter the behavior of the system at strategic locations in the block diagram of the multirate system facilitates discretization of the continuous-time subsystems and will also simplify the operations to be performed in Section 2.5. Before discretizing a continuous-time system, it is assumed that the following procedure is followed:

Procedure 2.2:

Assume that the multirate system in question satisfies Assumption 2.1 and that Procedure 2.1 has been performed. The subsystems referred to in the following steps are the subsystems which result from the partitioning required by Assumption 2.1(e).

a. Insert a sample and hold on each input and output of every discrete-time subsystem in the block diagram (points b,d,e,g, and i in Figure 2.3); its rate is that of the respective output or input of that subsystem.

b. Refer to both an output of a continuous-time subsystem and an input to the entire multirate system as a **(system) continuous output point**. For each *continuous output point in the* multirate system that is not immediately followed by a sampler (points a and c in Figure 2.3,) let $\{T_1,...,T_j\}$ be the set of periods of the first sampler encountered on each path leaving that continuous output point. Insert into the block diagram a sample and hold with period

$$T_J = \tau(\mathrm{GCD}\ \{p_1,...,p_j\}) = p_J\tau$$

immediately after that continuous output point.

c. Refer to both an input to a continuous-time subsystem and an output of the entire multirate system as a **(system) continuous input point**. For each continuous input point in the multirate system that is not immediately preceded by a sampler (points h and j in Figure 2.3,) let $\{T_1,...,T_j\}$ be the set of periods of the last sampler encountered on each path leading to that continuous input point. Insert into the block diagram a sample and hold with period

$$T_J = \tau(\mathrm{GCD}\ \{p_1,...,p_j\}) = p_J\tau$$

immediately before that continuous input point.

d. Refer to an output of a summer not in one of the subsystems composing the multirate system (points f, g, and j in Figure 2.3) as a **(connection) continuous output point**. Repeat the actions performed for system continuous

output points in step (b) for each connection continuous output point in the multirate system.

e. Refer to an input of a branch point not in one of the subsystems composing the multirate system (points a, e, and f in Figure 2.3) as a **(connection) continuous input point**. Repeat the actions performed for system continuous input points in step (c) for each connection continuous input point in the multirate system.

♦♦♦

The satisfaction of parts (e) and (f) of Assumption 2.1 assures the existence of the rates required in parts (b) and (c) of Procedure 2.2. All possible sources/destinations of signals in the multirate system are thus followed/preceded by samplers; consequently, the rates required in parts (d) and (e) of Procedure 2.2 will exist. The application of Procedure 2.2 does not alter the behavior of the system; each added sampler updates at least at the times that the samplers feeding from/to it update. By definition of T_J (see parts (b) and (c) of Procedure 2.2,) if a sampler with period T_i, $i \in \{1,...,j\}$, samples, the sampler with period T_J also samples: $t = nT_i$, $n \in \mathbb{Z}^+$ implies $t/T_J = np_i\tau/p_J\tau = n(p_i/p_J) \in \mathbb{Z}^+$.

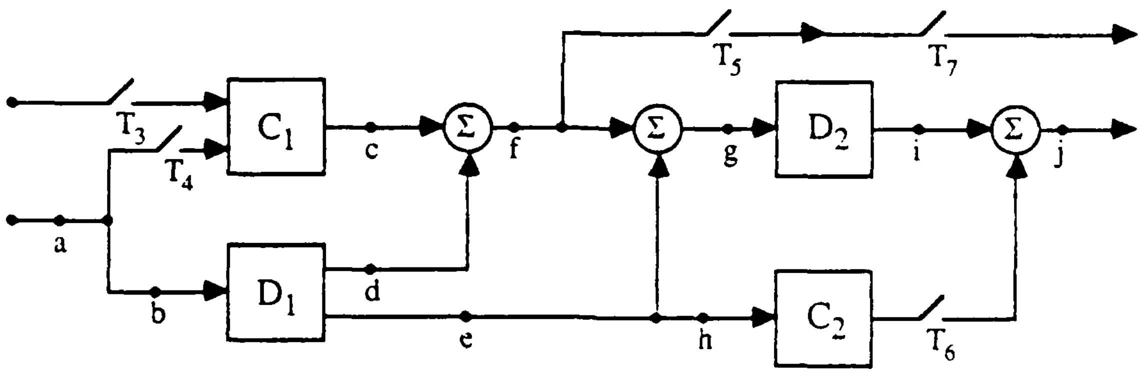

Figure 2.3. C_1 and C_2 are continuous-time subsystems, and D_1 and D_2 are discrete-time subsystems.

Example 2.3:

Let D_1 and D_2 in Figure 2.3 be single rate with rates $1/T_1$ and $1/T_2$, respectively. The application of Procedure 2.2 to the multirate system in Figure 2.3 proceeds as follows:

a: Insert samplers of period $T_b = T_d = T_e = T_1$ at points b,d, and e and samplers of period $T_g = T_i = T_2$ at points g and i.

b: Insert a sampler of period $T_a = \tau(GCD\ \{p_4, p_b\})$ at point a and a sampler of period $T_c = \tau(GCD\ \{p_5, p_g\})$ at point c.

c: Insert a sampler of period $T_j = \tau(GCD\ \{p_6, p_i\})$ at point j and a sampler of period $T_h = T_e$ at point h.

d: Insert a sampler of period $T_f = \tau(GCD\ \{p_5, p_g\})$ at point f. Points g and j are also connection continuous output points, but samplers are already present at these points.

e: Points a,e, and f are connection continuous input points, but samplers are already present at these points.

◆◆◆

After applying Procedure 2.2, each input and output line of a continuous-time subsystem on the block diagram should be designated as a signal bundle. The last sampler an input bundle passes through or the first sampler an output bundle passes through determines the rate assigned to that bundle during the discretization process.

Discretization of a multirate sampled continuous-time system is also considered in Araki and Yamamoto (1986), from a perspective quite different from the one taken here; Araki and Yamamoto (1986) employ expanded states, as well as expanded signals.

Let the continuous-time system

$$\dot{x} = A\,x + B\,u, \quad y = C\,x + D\,u \tag{2.4}$$

have the input

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

where each u_i is a signal bundle sampled and held at rate $1/T_i$, $i = 1,\dots,m$, and B is partitioned to be compatible with u: $B = [b_1\ b_2\ \dots\ b_m]$. Let y be partitioned into q bundles, with each bundle y_j sampled with period $\bar{T}_j$, $j = 1,\dots,q$; let C and D be partitioned accordingly:

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_q \end{bmatrix}, \text{ and } D = \begin{bmatrix} d_{11} & \dots & d_{1m} \\ \vdots & & \vdots \\ d_{q1} & \dots & d_{qm} \end{bmatrix}.$$

To discretize this system, let $L = LCP\ \{T_1,\dots,T_m,\bar{T}_1,\dots,\bar{T}_q\}$, $x(kL) \in \Re^n$ denote the state of (2.4) at time kL, and Y(kL) and U(kL) denote the L-expanded versions of y and u. Let

$$s_i = T/T_i, \;\; \bar{s}_j = T/\bar{T}_j, \;\; \Phi(w) = e^{Aw}, \text{ and } \Gamma(w) = \int_0^w e^{A(w-t)}dt\,.$$

Then a straightforward but tedious calculation gives

$$
\begin{aligned}
x((k+1)L) &= E\,x(kL) + F\,U(kL) \\
Y(kL) &= G\,x(kL) + H\,U(kL),
\end{aligned}
\tag{2.5}
$$

where

1. $E = \Phi(L)$.

2. $F = [f_1\ f_2 \ldots f_m]$, partitioned compatible with $U(kL)$, where

$$f_i = [\Phi(T_i(s_i-1))\Gamma(T_i)b_i \mid \ldots \mid \Phi(T_i)\Gamma(T_i)b_i \mid \Gamma(T_i)b_i]\ ,\ i = 1,\ldots,m.$$

3.

$$G = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_q \end{bmatrix} \text{ is partitioned as } Y(kL), \text{ where } g_j = \begin{bmatrix} c_j \\ c_j\Phi(\overline{T}_j) \\ \vdots \\ c_j\Phi(\overline{T}_j(\bar{s}_j-1)) \end{bmatrix},\ j = 1,\ldots,q.$$

4.

$$H = \begin{bmatrix} h_{11} & \cdots & h_{1m} \\ \vdots & & \vdots \\ h_{q1} & \cdots & h_{qm} \end{bmatrix}$$

is partitioned according to $Y(kL)$ and $U(kL)$, and each subblock h_{ij}, $i = 1,\ldots,q$, $j = 1,\ldots,m$, is composed of blocks $(h_{ij})_{rv}$, $r = 1,\ldots,\bar{s}_i$, $v = 1,\ldots,s_j$, each having the same dimension as d_{ij} and where

$$(h_{ij})_{rv} = \begin{cases} 0\ , & \text{if } (r-1)\overline{T}_i < (v-1)T_j \\ d_{ij} + c_i\Gamma((r-1)\overline{T}_i - (v-1)T_j)b_j\ , & \text{if } (v-1)T_j \leq (r-1)\overline{T}_i < vT_j\ . \\ c_i\Phi((r-1)\overline{T}_i - vT_j)\Gamma(T_j)b_j\ , & \text{if } vT_j \leq (r-1)\overline{T}_i \end{cases}$$

From (2.5) and points 1 through 4 above, the discretized multirate sampled continuous-time subsystem is a time-invariant single-rate discrete-time system with inputs and outputs that are vectors of L-expanded bundles. Thus, the methods in Section 2.3 apply to further expansion of the discretized continuous-time system.

Despite the formidable appearance of the discretized equations, the required calculations can be performed using any software package capable of discretizing a continuous-time state equation at a single rate. The quantity $\Gamma(w)B$ is the "B" matrix resulting from discretizing (A,B) at rate $1/w$. The required arguments of $\Phi(.)$ and $\Gamma(.)$ are all integer multiples of τ. For some values of i, j, r,

and v, the arguments of $\Phi(.)$ and $\Gamma(.)$ will coincide, so not every occurrence of $\Phi(.)$ and $\Gamma(.)$ in the above equations entails an additional calculation.

A modification to Procedure 2.2, which greatly simplifies the discretization of multirate sampled continuous-time subsystems, is to place sample/holds of period τ on the block diagram at the inputs and outputs of each continuous-time subsystem, leaving Procedure 2.2 otherwise unchanged. These added samplers leave the behavior of the system unaltered for the same reason as that given following Procedure 2.2. This modification has the advantage that discretization can be carried out uniformly at rate τ, but it may result in inputs and outputs of unacceptably large dimensions after expansion.

Example 2.4:

Let the continuous-time subsystem in Figure 2.4 have the state equation

$$\dot{x}(t) = A\, x(t) + B\, u(t)$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x(t) + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} u(t).$$

Additional samplers do not need to be added to discretize this system. Following the steps above, let

$$\Phi(w) = e^{Aw}, \text{ and } \Gamma(w) = \int_0^w e^{A(w-t)} dt\ .$$

Then

$$x((n+1)6\tau) = \Phi(6\tau)\, x(n(6\tau)) + [\Phi(3\tau)\Gamma(3\tau)B \quad \Gamma(3\tau)B] \begin{bmatrix} u(n(6\tau)) \\ u(n(6\tau)+3\tau) \end{bmatrix}$$

and

$$\begin{bmatrix} y_1(n(6\tau)) \\ y_1(n(6\tau)+\tau) \\ y_1(n(6\tau)+2\tau) \\ y_1(n(6\tau)+3\tau) \\ y_1(n(6\tau)+4\tau) \\ y_1(n(6\tau)+5\tau) \\ y_2(n(6\tau)) \\ y_2(n(6\tau)+2\tau) \\ y_2(n(6\tau)+4\tau) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_1\Phi(\tau) \\ c_1\Phi(2\tau) \\ c_1\Phi(3\tau) \\ c_1\Phi(4\tau) \\ c_1\Phi(5\tau) \\ c_2 \\ c_2\Phi(2\tau) \\ c_2\Phi(4\tau) \end{bmatrix} x(n(6\tau)) + \begin{bmatrix} d_1 & 0 \\ d_1+c_1\Gamma(\tau)B & 0 \\ d_1+c_1\Gamma(2\tau)B & 0 \\ c_1\Gamma(3\tau)B & d_1 \\ c_1\Phi(\tau)\Gamma(3\tau)B & d_1+c_1\Gamma(\tau)B \\ c_1\Phi(2\tau)\Gamma(3\tau)B & d_1+c_1\Gamma(2\tau)B \\ d_2 & 0 \\ d_2+c_2\Gamma(2\tau)B & 0 \\ c_2\Phi(\tau)\Gamma(3\tau)B & d_2+c_2\Gamma(\tau)B \end{bmatrix} \begin{bmatrix} u(n(6\tau)) \\ u(n(6\tau)+3\tau) \end{bmatrix}.$$

Note that all of the quantities above could be obtained by normal discretization with periods τ, 2τ, and 3τ. Also, if the modification to Procedure 2.2 noted above were

used, the resulting discretized system would have 6 inputs and 12 outputs.

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Figure 2.4. A multirate sampled continuous-time subsystem.

2.5 Connection Matrices

The concept of a connection matrix facilitates the interconnection of subsystems whose inputs and outputs are expanded bundles of signals which pass through summers, samplers, and branch points between the subsystems. An individual connection matrix serves to connect a signal bundle at one rate to a signal bundle at another rate after expansion. The form of the connection matrix depends only on the rates of the bundles being connected and the amount of time over which the signals were expanded.

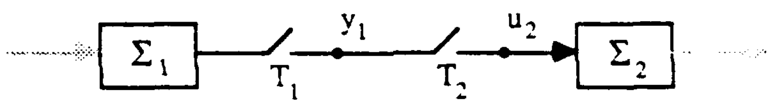

Figure 2.5. A direct connection between subsystems.

Consider a single-rate subsystem, Σ_1, with a bundle of r outputs connected directly to a single-rate subsystem with a bundle of r inputs, Σ_2, as shown in Figure 2.5. Let $L/T_1 = l_1$, $L/T_2 = l_2$, and $l_1, l_2 \in \mathbb{N}$. Since $l_1 \neq l_2$ in general, the number of outputs of the L-expanded description of Σ_1 will not equal the number of inputs of the L-expanded description of Σ_2. The mechanism which serves to connect $Y_1(kL)$ to $U_2(kL)$ is the sampling Σ_2 performs on its input:

$$u_2(kL+(i-1)T_2) = y_1(kL+(j-1)T_1) \ \text{ if } \ (j-1)T_1 \leq (i-1)T_2 < jT_1 ,$$

where $i \in \{1,...,l_2\}$ and $j \in \{1,...,l_1\}$. This relationship can be expressed concisely by defining $Q(L,r,1{:}2)$, the **connection matrix** for L-expanded bundles of r signals from period T_1 to period T_2, such that $U_2(kL) = Q(L,r,1{:}2)Y_1(kL)$. $Q(L,r,1{:}2)$ is an $rl_2 \times rl_1$ matrix of $l_2 \times l_1$ $r \times r$ blocks, where the i,j^th block is given by

$$Q(L,r,1{:}2)_{ij} = \begin{cases} I_r , & \text{if } \ (j-1)p_1 \leq (i-1)p_2 < jp_1 \\ 0_r , & \text{otherwise} \end{cases} .$$

Note that if $l_1 = l_2 = l_0$, then $Q(L,r,1{:}2) = I_{rl_0}$.

Example 2.5:

Let $T_2 = 3\tau$, $T_3 = 5\tau$, and $L = 30\tau$. Then

$$Q(L,1,2:3) = \begin{bmatrix} 10000 & 00000 \\ 01000 & 00000 \\ 00010 & 00000 \\ 00000 & 10000 \\ 00000 & 01000 \\ 00000 & 00010 \end{bmatrix} \text{ and } Q(L,1,3:2) = \begin{bmatrix} 100 & 000 \\ 100 & 000 \\ 010 & 000 \\ 010 & 000 \\ 001 & 000 \\ 000 & 100 \\ 000 & 100 \\ 000 & 010 \\ 000 & 010 \\ 000 & 001 \end{bmatrix}.$$

♦♦♦

A fact which is easily proven by induction is that if $M/T_1 \in \mathbb{N}$ and $M/T_2 \in \mathbb{N}$, then for some $m \in \mathbb{N}$, $M = m\tau(\text{LCM }\{p_1,p_2\})$. In words, any M admissible for expanding signals with periods T_1 and T_2 is an integer multiple of the least common period of T_1 and T_2. Let $T(1,2) = \tau(\text{LCM }\{p_1,p_2\})$. Since $U_2(nT(1,2)) = Q(T(1,2),r,1:2)Y_1(nT(1,2))$, this connection matrix may be thought of as a discrete-time system with rate $1/T(1,2)$, input $Y_1(nT(1,2))$, output $U_2(nT(1,2))$, and representation $(0,0,0,Q(T(1,2),r,1:2))$. After finding the M-expanded representation for this system,

$$U_2(kM) = \text{diag}[Q(T(1,2),r,1:2),Q(T(1,2),r,1:2),\ldots,Q(T(1,2),r,1:2)]\ Y_1(kM);$$

$Q(T(1,2),r,1:2)$ appears m times along the diagonal. Thus,

$$Q(M,r,1:2) = \text{diag}[Q(T(1,2),r,1:2),Q(T(1,2),r,1:2),\ldots,Q(T(1,2),r,1:2)].$$

This property greatly reduces the effort required to compute and store connection matrices for large m. In Example 2.5, m equals 2.

Connection matrices can be systematically placed so that a collection of L-expanded subsystems is connected according to a given block diagram. Since separate inputs and outputs drawn on a block diagram were defined as distinct bundles of signals, connection matrices can effect direct connections between subsystems. Under the assumption that Procedure 2.2 has been applied to the block diagram, rules for placing connection matrices when the bundles of signals pass through samplers, summers, and branch points between the subsystems are easily stated.

Define a **link** as any path from one sampler to another on the block diagram which may pass through summers and branch points but not through subsystems or other samplers. As a result of applying Procedure 2.2, in the region between the subsystems a sampler follows each summer and precedes each branch point. Thus, a link passes through at most one summer and one

branch point. The steps necessary to connect a collection of L-expanded subsystems together according to a given block diagram are detailed below.

Procedure 2.3:

Assume that all subsystems and signals in the block diagram are in L-expanded form and that Procedure 2.2 has been applied to the block diagram. For each link in the system to be connected:

a. Find Q(L,r,f:t), where the r signals on the link flow from a sampler with period T_f to a sampler with period T_t.

b. Insert the connection matrix in the link according to the type of link as follows:

i. If the link traverses a branch point and a summer, place the connection matrix after the branch point and before the summer.

ii. If the link crosses only a branch point, place the connection matrix after the branch point.

iii. If the link traverses only a summer, insert the connection matrix before the summer.

iv. If none of the above apply, the link crosses no summers or branch points, and the connection matrix can be placed anywhere on the link.

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As an aid to justifying the connection matrix locations specified in Procedure 2.3, note that their placement ensures that the dimensions of all inputs of each summer and of all outputs of each branch point in the block diagram of the expanded system are identical.

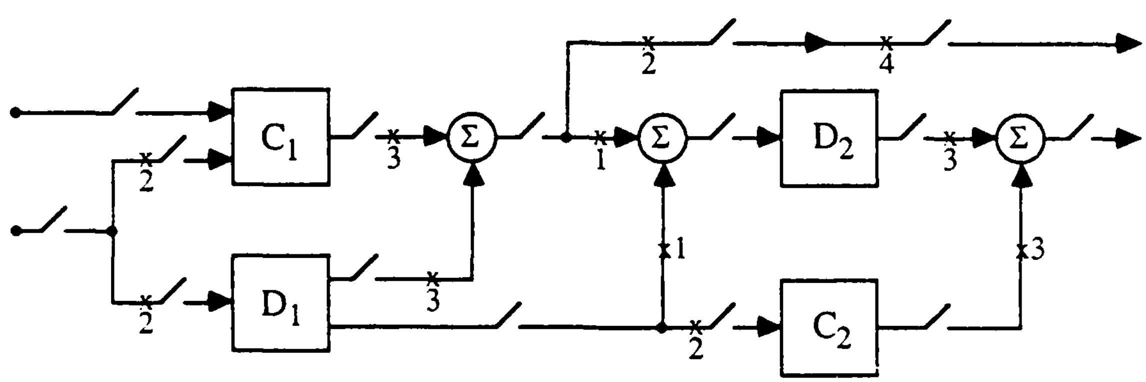

Figure 2.6. Placement of connection matrices.

Example 2.6:

Figure 2.6 shows the placement of connection matrices in the multirate system in Figure 2.3 after Procedure 2.2 has been performed. The numbers 1, 2, 3, and 4 in Figure 2.6 denote a link of type i, ii, iii, or iv, respectively, as described in Procedure 2.3(b).

♦♦♦

2.6 The T-expanded Representation of a Multirate System

The preceding sections collectively provide a means of obtaining a time-invariant input-output or state space description of a multirate system, which will be referred to as the **T-expanded representation**. The reader may wish to keep the following general procedure in mind as a summary of the previous sections.

Procedure 2.4:

a. Starting with a block diagram of the system, verify that Assumption 2.1 is satisfied. The partitioning required in Assumption 2.1(e) may entail a trial and error approach. For the remaining steps of this procedure, "subsystem" refers *specifically to the subsystems obtained as a result of this partitioning*.

b. Assign labels to all sampling periods in the system, and apply Procedure 2.1 to find T.

c. Find a state space realization for each continuous-time and discrete-time subsystem.

d. Apply Procedure 2.2 to the block diagram. Procedure 2.2(b) may be performed in parallel with Procedure 2.2(c), and Procedure 2.2(d) and Procedure 2.2(e) may be performed in parallel. However, do not reverse the order of these two pairs of steps.

e. Identify the rates at which the inputs and outputs of continuous-time subsystems are sampled and discretize these subsystems as described in Section 2.4.

f. Designate lines on the block diagram representing inputs and outputs of subsystems and inputs and outputs of the entire multirate system as bundles of signals. Partition the state space realization of each subsystem accordingly.

g. Find the T-expanded representation of each subsystem as detailed in Section 2.3.

h. Apply Procedure 2.3 and then delete all of the samplers from the block diagram.

i. Note that the block diagram obtained in step (h) is a block diagram of interconnected time-invariant discrete-time systems with period T. The states at times kT of the subsystems in this block diagram and the input-output behavior of the entire block diagram as a system are identical to those of the original multirate system in step (c) (after decomposing the expanded signals into their components.) Standard methods may now be applied to reduce this block diagram to a single state space equation or transfer function matrix.

♦♦♦

Completion of Procedure 2.4(i) will in general involve a matrix inversion to find the state space description of a feedback structure. Under certain conditions, this inverse may not exist; hence, a state space representation for such a feedback structure cannot be found. As an example, the seemingly innocuous system in Figure 2.7 composed of two samplers and two continuous-time unity gain blocks has no state space description for any choices of T_1 and T_2. This difficulty is intimately related to the well-posedness problem encountered in transfer function descriptions of composite systems. See Chen (1984) for a discussion of this problem. If the multirate system as modeled by the block diagram in Procedure 2.4(a) has the property that each closed path on the block diagram passes through at least one subsystem which has no direct feedthrough, then it is suspected that the difficulties described above will not be encountered.

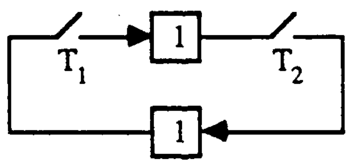

Figure 2.7. An ill-posed multirate system.

2.7 Inclusion of Periodic Subsystems

A slight extension of Procedure 2.4 permits the inclusion of single-rate periodic discrete-time subsystems in the multirate system. Consider a single-rate discrete-time subsystem with single-bundle inputs and outputs and a realization which is time varying, but periodic. For all $n \in \mathbb{Z}^+$, let

$$\Sigma((n+r)T_1) = \Sigma(nT_1) = (A(nT_1),B(nT_1),C(nT_1),D(nT_1)).$$

Thus, $\Sigma(nT_1)$ is periodic with period rT_1. As demonstrated in Meyer and Burrus (1975), repeated

iteration from an initial time of $t = 0$ of the state equation governing $\Sigma(nT_1)$,

$$x((n+1)T_1) = A(nT_1)\,x(nT_1) + B(nT_1)\,u(nT_1)$$
$$y(nT_1) = C(nT_1)\,x(nT_1) + D(nT_1)\,u(nT_1),$$

yields a time-invariant state equation with rate $1/rT_1$:

$$x((k+1)rT_1) = A_e\,x(k(rT_1)) + B_e\,U(k(rT_1))$$
$$Y(k(rT_1)) = C_e\,x(k(rT_1)) + D_e\,U(k(rT_1)).$$

Denoting $A(nT_1)$, $B(nT_1)$, etc. as $A(n)$, $B(n)$, etc. for brevity,

$$A_e = A(r-1)A(r-2)...A(1)A(0),$$

$$B_e = [A(r-1)...A(1)B(0) \mid ... \mid A(r-1)A(r-2)B(r-3) \mid A(r-1)B(r-2) \mid B(r-1)],$$

$$C_e = \begin{bmatrix} C(0) \\ C(1)A(0) \\ C(2)A(1)A(0) \\ \vdots \\ C(r-1)A(r-2)...A(0) \end{bmatrix},$$

and D_e is composed of $r \times r$ of the blocks d_{ij}, where

$$d_{ij} = \begin{cases} 0\ , & \text{if } i < j \\ D(i-1)\ , & \text{if } i = j \\ C(i-1)B(j-1)\ , & \text{if } i = j+1 \\ C(i-1)A(i-2)...A(j)B(j-1)\ , & \text{if } i > j+1 \end{cases} \qquad (2.6)$$

Thus, (A_e, B_e, C_e, D_e) serves as a single-bundle rT_1-expanded representation of $\Sigma(nT_1)$. This time-invariant representation can be further expanded using the technique in Section 2.3. If Procedure 2.4 were applied to a multirate system containing this periodic subsystem, the only alteration of Procedure 2.4 required would be to include both T_1 and rT_1 in Procedure 2.4(b) to find T. In analogy with Section 2.3, the expanded representation of a periodic discrete-time subsystem $\underline{\Sigma}(nT_1)$ with inputs $\underline{u}(nT_1)$ and outputs $\underline{y}(nT_1)$ that are vectors of bundles can be expressed in terms of single-bundle expanded representations by first partitioning $\underline{B}(nT_1)$, $\underline{C}(nT_1)$, and $\underline{D}(nT_1)$ to conform with the bundles composing $\underline{u}(nT_1)$ and $\underline{y}(nT_1)$.

A multirate periodic discrete-time subsystem may be **specified in T_1-expanded form** by both specifying $A(kT_1)$, $B(kT_1)$, $C(kT_1)$, and $D(kT_1)$ in its state equation,

$$x((k+1)T_1) = A(kT_1)\,x(kT_1) + B(kT_1)\,U(kT_1)$$
$$Y(kT_1) = C(kT_1)\,x(kT_1) + D(kT_1)\,U(kT_1), \qquad (2.7)$$

where $Y(kT_1)$ and $U(kT_1)$ are T_1-expanded versions of vectors of bundles of signals and $A(kT_1)$, $B(kT_1)$, $C(kT_1)$, and $D(kT_1)$ are rT_1-periodic, and by specifying the partition of $Y(kT_1)$ and $U(kT_1)$ into expanded bundles and the rates associated with each expanded bundle. Although discretization of a rT_1-periodic continuous-time system with inputs and outputs sampled at multiple rates would yield a state equation such as (2.7), the state equation (2.7) may not be the result of discretizing or expanding any subsystem. In the case $r = 1$, (2.7) may represent the behavior of a computer program performing concurrent tasks at multiple rates. In Chapter 5, the controllers designed will take the form of (2.7). The subsystem (2.7) may be expanded over its period, rT_1, and then further expanded over time T during the execution of Procedure 2.4.

The point of the extensions of Procedure 2.4 given above is not to belabor specific examples but to indicate a general property of Procedure 2.4. If, by some means, a time-invariant discrete-time description of a subsystem in terms of expanded inputs and outputs can be obtained, Procedure 2.4 applies to a multirate system in which this subsystem appears as a component. In this instance, Procedure 2.4 must only be modified by including the period of the time-invariant description of this subsystem in the set of sampling periods considered when finding the least common period of the entire multirate system, T.

Example 2.7:

Suppose that a T-expanded representation is desired for the multirate system shown in Figure 2.8, where $T_1 = 1$ *sec*, $T_2 = 0.5$ *sec*, *and* $g(nT_1)$ is a discrete-time periodic gain with period $T_3 = 2T_1$:

$$g(nT_1) = \begin{cases} g_1 \text{ , if n is even} \\ g_2 \text{ , if n is odd} \end{cases} .$$

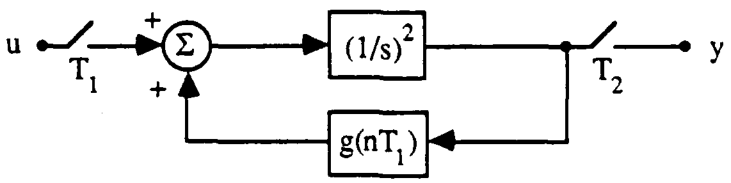

Figure 2.8. A multirate system.

Applying Procedure 2.4:

a: Considering the double integrator and the discrete-time gain as subsystems, parts (a), (e), and (f) of Assumption 2.1 are satisfied. It will be assumed that the rest of Assumption 2.1 is satisfied as well (with the exception that a periodic discrete-time subsystem is present.)

b: Application of Procedure 2.1 yields $\tau = 0.5$ sec, $p_1 = 2$, $p_2 = 1$, $p_3 = 4$, and $T = 2$ sec.

c: A state space realization for the periodic gain is $(0,0,0,g(nT_1))$, and a state space realization for the double integrator is

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [\ 1\ \ 0\ ],\ 0 \right).$$

d: If the modified version of Procedure 2.2 (wherein continuous-time subsystems are sampled with period τ (= T_2)) is applied to the system, Figure 2.9 results.

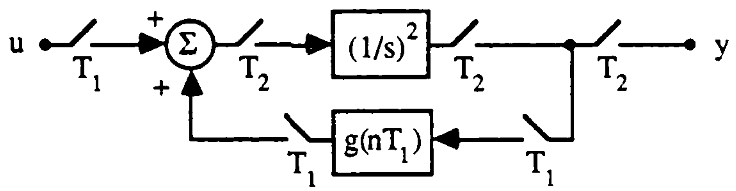

Figure 2.9. The system in Figure 2.8 with inserted samplers.

e: Discretizing the double integrator with period T_2 gives

$$A_d = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix},\ B_d = \begin{bmatrix} 1/8 \\ 1/2 \end{bmatrix},\ C_d = [\ 1\ 0\ ]\text{, and } D_d = 0.$$

f: This part is trivial since all inputs and outputs are drawn as single lines.

g: Denote the T-expanded representation of the periodic gain by K. From (2.6),

$$K = \begin{bmatrix} g_1 & 0 \\ 0 & g_2 \end{bmatrix}.$$

The T-expanded representation of (A_d,B_d,C_d,D_d) can be found using (2.2) with $i_2 = T/T_2 = 4$:

$$A_e = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},\ B_e = \begin{bmatrix} 7/8 & 5/8 & 3/8 & 1/8 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix},$$

$$C_e = \begin{bmatrix} 1 & 0 \\ 1 & 1/2 \\ 1 & 1 \\ 1 & 3/2 \end{bmatrix}\text{, and } D_e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1/8 & 0 & 0 & 0 \\ 3/8 & 1/8 & 0 & 0 \\ 5/8 & 3/8 & 1/8 & 0 \end{bmatrix}.$$

h: Let $Q_1 = Q(T,1,1{:}2)$, $Q_2 = Q(T,1,2{:}1)$, and $Q_3 = Q(T,1,2{:}2)$. Then

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ and } Q_3 = I_4.$$

After inserting connection matrices and removing samplers, the block diagram of the T-expanded representation of the system appears as in Figure 2.10.

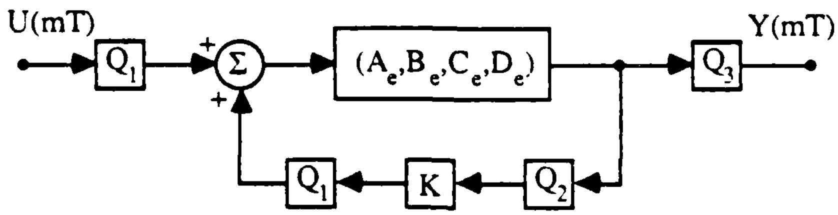

Figure 2.10. Block diagram of the T-expanded representation.

i: The system in Figure 2.10 can be reduced to a single state space equation, the T-expanded representation of the system in Figure 2.8:

$$x((m+1)T) = \begin{bmatrix} \frac{4+6g_1+g_1g_2+2g_2}{4} & \frac{4+g_2}{2} \\ \frac{g_1g_2+2g_1+2g_2}{2} & 1+g_2 \end{bmatrix} x(mT) + \begin{bmatrix} \frac{6+g_2}{4} & \frac{1}{2} \\ \frac{2+g_2}{2} & 1 \end{bmatrix} \begin{bmatrix} u(mT) \\ u(mT+T_1) \end{bmatrix}$$

$$\begin{bmatrix} y(mT) \\ y(mT+T_2) \\ y(mT+2T_2) \\ y(mT+3T_2) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1/2 \\ 1 & 1 \\ 1 & 3/2 \end{bmatrix} x(mT) + \begin{bmatrix} 0 & 0 \\ 1/8 & 0 \\ 1/2 & 0 \\ 1 & 1/8 \end{bmatrix} \begin{bmatrix} u(mT) \\ u(mT+T_1) \end{bmatrix}.$$

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2.8 Modular Expansion of Multirate Systems

A fundamental difficulty encountered in executing Procedure 2.4 is that the T-expanded representation of the subsystems may involve matrices of large dimensions. A simple example demonstrates this problem and suggests a method of circumventing it, referred to as **modular expansion.**

Consider the multirate system with SISO time-invariant subsystems in Figure 2.11, where $T_1 = 2$, $T_2 = 1$, $T_3 = 25$, and $T = 50$. Straightforward application of Procedure 2.4 yields a

T-expanded representation for Σ_1 with 25 inputs and outputs and a T-expanded representation for Σ_2 with 50 inputs and outputs. Reduction of the feedback connection of the expanded representations to a single state equation involves the multiplication and addition of large matrices and, more importantly, the inversion of two $\underline{25 \times 25}$ matrices.

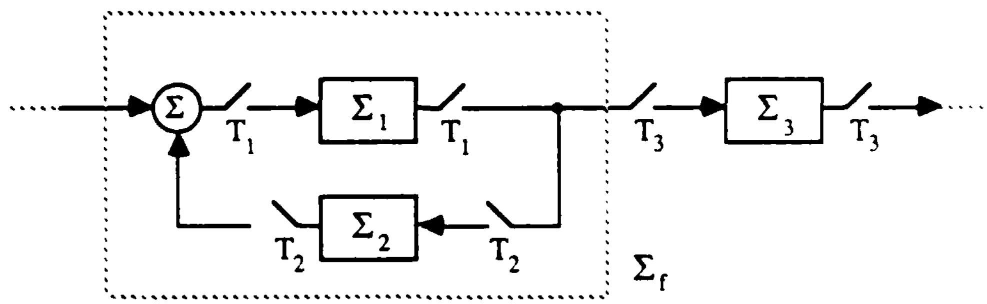

Figure 2.11. A simple multirate system.

As an alternative method, consider ignoring Σ_3 for the moment and finding an expanded representation for the feedback connection of Σ_1 and Σ_2 over the least common period of T_1 and T_2 using Procedure 2.4. Let $T_f = \mathrm{LCP}\ \{T_1, T_2\} = 2$. Calculation of the T_f-expanded representation for the feedback connection involves much smaller matrices and, in particular, requires the inversion of two $\underline{1 \times 1}$ matrices. This savings in computation will be worthwhile if the T-expanded representation of Σ_f exists, enabling the T-expanded representations of Σ_f and Σ_3 to be combined. A T_f-expanded representation can be T-expanded if $T/T_f \in \mathbb{N}$. Since Theorem 2.3 asserts that $T/T_f = \mathrm{LCP}\ \{T_1, T_2, T_3\}/\mathrm{LCP}\ \{T_1, T_2\} \in \mathbb{N}$, the T-expanded representation of Σ_f can be computed. This then is the essence of modular expansion.

There is substantial freedom in the steps taken when performing modular expansion on a multirate system. The following procedure helps explain the notion of modular expansion and should not be interpreted as the only means by which modular expansion can be performed.

Procedure 2.5: (a modular expansion scheme)

a. Carry out steps (a) through (f) of Procedure 2.4.

b. Focus attention on a collection of subsystems by drawing a path on the block diagram enclosing one or more subsystems which does not separate any enclosed continuous output/input point from the sampler immediately following/preceding it (review Procedure 2.2 for the meaning of these terms.) Call the enclosed portion of the system the "current collection." Set $i = 0$ and $P(0) = 1$. Let $\{T_{0,1}, T_{0,2}, \ldots, T_{0,m(0)}\}$ be the set of $m(0)$ sampling periods and periodic discrete-time subsystem periods in the current collection.

c. Set $P(i+1) = \mathrm{LCM}\ \{P(i), p_{i,1},\ldots,p_{i,m(i)}\}$ and $T_{(i+1),0} = P(i+1)\tau$. Find the $T_{(i+1),0}$-expanded representation of each subsystem in the current collection and insert connection matrices. By standard techniques, find a single discrete-time state equation with rate $1/T_{(i+1),0}$ to describe the current collection. When finding this state equation, ignore samplers on the block diagram that are between connection matrices, subsystems, summers, or branch points in the current collection but retain samplers on the periphery of the current collection.

d. Set $i = i+1$.

e. If the current collection is the entire multirate system, stop. Otherwise, draw a path enclosing the current collection and possibly other subsystems, following the same restrictions as in step (b) above. Let $\{T_{i,1},T_{i,2},\ldots,T_{i,m(i)}\}$ be the m(i) additional sampling periods and periodic discrete-time subsystem periods enclosed by this path. Call the portion of the system enclosed by this path the current collection and return to step (c).

♦♦♦

Procedure 2.5 proceeds by repeatedly adding subsystems to a single collection of subsystems. Modular expansion may in general start with several collections scattered throughout the block diagram and repeatedly add subsystems and collections of subsystems to these collections. The restrictions in Procedure 2.5(b) on the items included in a collection must be obeyed during the process of enlarging a collection. Each time a new collection is formed, that collection is expanded and then reduced to a single state equation as in Procedure 2.5(c). The process terminates when a single collection contains the entire multirate system. The principle that allows collections of subsystems to be expanded and combined in such a variety of ways is that, as a simple induction argument and Theorem 2.3 show, the least common period of the union of a collection of sets of sampling periods is an integer multiple of the least common period of each set in that collection.

A trade-off between the effort required to combine expanded subsystems and the effort expended in expanding a representation corresponding to a combination of subsystems complicates the question of which modular expansion strategy results in the least computational effort for a given multirate system. Two general heuristics for modular expansion can be offered. To avoid inverting large matrices, expand subsystems in a feedback connection and obtain a representation describing this feedback configuration before including it in a larger collection of subsystems. By choosing collections of subsystems that have fewer input and output lines, the dimensions of the required "B", "C", and "D" matrices are reduced.

CHAPTER 3

PERIODIC REPRESENTATION OF MULTIRATE SYSTEMS

Representing the behavior of a multirate discrete-time system with period T over intervals of time of length M, where $T/M \in \mathbb{Q}$, results in a description which is time varying and periodic. Such a description will be referred to as the **M-varying representation** of the system. The motivation for examining the M-varying representation is twofold. Not only does it provide a periodic representation of the multirate system, but with $M = \tau$, the M-varying representation finds use theoretically in investigating the interperiod behavior of a multirate system by examining its T-expanded representation. Three intrinsic difficulties detract from the practical usefulness of the M-varying representation presented here. In general, the M-varying representation has more states than the time-invariant description, is difficult to calculate, and involves padding with false inputs and outputs to prevent the number of inputs and outputs from varying with time.

Many possibilities exist for the periodic representation of a multirate system. In the M-varying representation, state transitions occur at regular intervals. A representation like the one presented in Kalman and Bertram (1959) in which state transitions occur only at the times of certain events, such as a sampling event or a state transition of a discrete-time element, is perhaps more appealing from a practical point of view due to reduced storage requirements. However, the M-varying representations presented here are valid periodic representations of the multirate system and are easily described in detail.

3.1 The τ-varying Representation

In Chapter 2, the system was represented over an interval of time which was an integer multiple of the period of each sampler. Inputs and outputs were implicitly "held" by retaining their values as components of time expanded signals. The τ-varying representation of a system does not permit this luxury, and the values stored in most hold circuits must be retained as state variables.

Consider a sample/hold circuit with period $T_1 = p_1\tau$, input u, and output y. Let $x(n\tau)$ be the value stored in the hold circuit at time $t = n\tau$. For times $t = n_0\tau$ at which sampling occurs, the next state is updated; as a consequence of Assumption 2.1(d), the output is set to the input:

$$x((n_0+1)\tau) = u(n_0\tau) \text{ and } y(n_0\tau) = u(n_0\tau).$$

At all other times, the state remains unchanged and the output is set to the value of the state:

$$x((n+1)\tau) = x(n\tau) \text{ and } y(n\tau) = x(n\tau).$$

Thus, for a sample/hold with period T_1,

$$x((n+1)\tau) = a(n\tau)\, x(n\tau) + b(n\tau)\, u(n\tau)$$
$$y(n\tau) = c(n\tau)\, x(n\tau) + d(n\tau)\, u(n\tau),$$

where

$$(a,b,c,d)(n\tau) = \begin{cases} (0,1,0,1)\ , \text{ if } n \bmod p_1 = 0 \\ (1,0,1,0)\ , \text{ if } n \bmod p_1 \neq 0 \end{cases}.$$

Note that if $p_1 = 1$, the representation becomes time invariant, with corresponding state and output equations $x((n+1)\tau) = u(n\tau)$, $y(n\tau) = u(n\tau)$. Clearly, $x(n\tau)$ is superfluous in this case, and states need not be assigned to sample/holds of period τ.

The following convention is explicitly stated for clarity.

Convention 3.1: (Intermediate state values of discrete-time systems)

Given a single-rate discrete-time subsystem with state x and state transitions specified at times kT_1 or a multirate discrete-time subsystem with state x specified in T_1-expanded form, define

$$x(t) = x(kT_1),\ \ kT_1 \le t < ((k+1)T_1).$$

♦♦♦

Due to the multirate nature of the surrounding elements, a T_1-varying discrete-time subsystem must sample its inputs at times $t = kT_1$ and maintain its output at $y(t) = y(kT_1)$ for $kT_1 \le t < (k+1)T_1$. Since $u(kT_1)$ is required at time $t = kT_1+(p_1-1)\tau$ to determine the state transition, a state variable must be defined to retain $u(kT_1)$. Let a periodic T_1-varying single-rate discrete-time subsystem with realization $(A(kT_1),B(kT_1),C(kT_1),D(kT_1))$ have state $x \in \mathbb{R}^n$, input $u \in \mathbb{R}^m$ and output $y(kT_1)$. Define the augmented state

$$z((kp_1+i)\tau) = \begin{bmatrix} x((kp_1+i)\tau) \\ u(kT_1) \end{bmatrix},\ 0 < i \le p_1,\ k = 0,1,\ldots.$$

A τ-varying representation of this system which satisfies the state and output equations

$$z(t+\tau) = a(t)\, z(t) + b(t)\, u(t)$$
$$y(t) = c(t)\, z(t) + d(t)\, u(t)$$

for times $t = (kp_1+i)\tau$, $0 < i \le p_1$ and $k = 0,1,\ldots$, is

$$
(a,b,c,d)((kp_1+i)\tau) = \begin{cases} \left(\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \end{bmatrix}, [C(kT_1) \;\; 0], D(kT_1) \right), & \text{if } i = 0 \\ \left(\begin{bmatrix} I_n & 0 \\ 0 & I_m \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [C(kT_1) \;\; D(kT_1)], 0 \right), & \text{if } 0 < i < p_1 - 1 \\ \left(\begin{bmatrix} A(kT_1) & B(kT_1) \\ 0 & I_m \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, [C(kT_1) \;\; D(kT_1)], 0 \right), & \text{if } i = p_1 - 1 \end{cases} \tag{3.1}
$$

The reader may wish to trace through the above representation and note the following facts. $u(kT_1)$ is updated at the time it is available, $i = 0$, and the following output values and state transition are determined from this stored value. The output is held at $y((kp_1+i)\tau) = y(kT_1)$ for $0 \leq i < p_1$. This τ-varying representation is equivalent to the T_1-varying representation in the sense that for any $z(0) \in \mathbb{R}^{n+m}$ and input sequence $\{u(j\tau): j = 0,1,\ldots\}$, the resulting output sequence $\{y(j\tau)\}$ and the x portion of the state trajectory $\{z(j\tau)\}$ of the τ-varying representation are identical to samples at times $t = j\tau$ of the output sequence $\{y(kT_1)\}$ and the state trajectory $\{x(kT_1)\}$, where $x(j\tau)$ is obtained from $x(kT_1)$ via Convention 3.1, of the T_1-varying representation with initial state $x(0)$ and input $u = \{u(j\tau)\}$.

A periodic multirate discrete-time system specified in T_1-expanded form,

$$
\begin{aligned}
x(kT_1) &= A(kT_1)\, x(kT_1) + B(kT_1)\, U(kT_1) \\
Y(kT_1) &= C(kT_1)\, x(kT_1) + D(kT_1)\, U(kT_1),
\end{aligned}
$$

may also be expressed as a τ-varying system. It is necessary that $D(kT_1)$ be structured so that this system is causal. The state of the system must be augmented by $U(kT_1)$. Although the specific details of a τ-varying representation for a general multirate discrete-time system specified in time expanded form are too involved to present here, the basic philosophy is the same as that for a single-rate system. The values comprising $U(kT_1)$ are loaded into the augmented state as they become available, and the state transition occurs during the last τ interval of each T_1 interval, as specified in Convention 3.1. Strictly as a matter of convenience for developments in Section 3.3, the following convention should be satisfied by a τ-varying representation of a multirate discrete-time system specified in time expanded form.

Convention 3.2:

Portions of the augmented state corresponding to input values that are not yet available are assigned the value zero.

♦♦♦

Example 3.1:

A SISO multirate system with a single state is specified in 6τ-expanded

form:

$$x((k+1)6\tau) = a\, x(k6\tau) + [b_1\ b_2]\ U(k6\tau)$$

$$Y(k6\tau) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} x(k6\tau) + \begin{bmatrix} d_1 & 0 \\ d_2 & 0 \\ d_3 & d_4 \end{bmatrix} U(k6\tau).$$

The input u is sampled with period 3τ and the output y has period 2τ:

$$U(k6\tau) = \begin{bmatrix} u(k6\tau) \\ u(k6\tau+3\tau) \end{bmatrix},\ Y(k6\tau) = \begin{bmatrix} y(k6\tau) \\ y(k6\tau+2\tau) \\ y(k6\tau+4\tau) \end{bmatrix}.$$

Note the two 0's in the D matrix required for causality. Define the 3×1 augmented state z:

$$z((k6+i)\tau) = \begin{bmatrix} x((k6+i)\tau) \\ U(k6\tau) \end{bmatrix}, \quad 0 < i \le 6,\ k = 0,1,\ldots .$$

As this system has period 6τ, six sets of matrices are required for the τ-varying representation. These matrices are given as a(t), b(t), c(t), and d(t) for each t below.

$$t = k6\tau: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, [c_1\ 0\ 0], d_1$$

$$t = k6\tau + \tau: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, [c_1\ d_1\ 0], 0$$

$$t = k6\tau + 2\tau: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, [c_2\ d_2\ 0], 0$$

$$t = k6\tau + 3\tau: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, [c_2\ d_2\ 0], 0$$

$$t = k6\tau + 4\tau: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, [c_3\ d_3\ d_4], 0$$

$$t = k6\tau + 5\tau : \begin{bmatrix} a & b_1 & b_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, [c_3\ d_3\ d_4], 0 .$$

♦♦♦

A continuous-time subsystem whose inputs and outputs are sampled and held at multiple rates is readily converted to τ-varying form. The continuous-time subsystem is merely discretized with period τ and the surrounding sample/hold devices govern the flow of input and output values to and from the multirate sampled subsystem. The following procedure details the calculation of the τ-varying representation of a multirate discrete-time system.

Procedure 3.1:

a. Verify that Assumption 2.1 is met, with the exception that periodic single-rate and multirate discrete-time subsystems may be present. Partition the entire system into subsystems which satisfy Assumption 2.1(e).

b. Use Procedure 2.1 to normalize all sampling periods and periods of periodic subsystems and find $T = P\tau$.

c. Find a state space realization for each continuous-time subsystem and single-rate discrete-time subsystem.

d. Discretize with period τ each continuous-time subsystem and find a τ-varying representation for each sample/hold device and discrete-time subsystem in accordance with Convention 3.1 and Convention 3.2 as described in this section.

e. For each $j \in \{0,1,\ldots,P-1\}$, associate the $t = j\tau$ value of the τ-varying representation of each subsystem and sample/hold device with the corresponding subsystems and sample/hold devices in the block diagram of the system. Reduce this block diagram to a single state space equation as if it were the block diagram of a single-rate system. This gives a T-periodic, τ-varying representation for the entire multirate system: $(A(n\tau),B(n\tau),C(n\tau),D(n\tau))$.

♦♦♦

As with Procedure 2.4, difficulties may arise when performing Procedure 3.1(e).

In certain instances, Procedure 3.1(d) can be modified in order to reduce the number of states of the τ-varying representation. For example, if a signal is connected to a sample/hold with period T_1 or to a single-rate discrete-time subsystem which is T_1-sampling the signal and this signal is the output of a sample/hold with period T_2, a T_2-held output of a discrete-time system, or the sum of such signals and $T_2/T_1 \in \mathbb{N}$, then a state variable need not be assigned to the sample/hold with period T_1 or to retain the T_1-sampled input value of the discrete-time subsystem. In this case, a τ-varying representation for a T_1-varying single-rate discrete-time subsystem is, in

the notation of equation (3.1),

$$(a,b,c,d)((kp_1+i)\tau) = \begin{cases} (I_n,\ 0,\ C(kT_1),\ D(kT_1)) \ , \text{ if } 0 \le i < p_1 - 1 \\ (A(kT_1),\ B(kT_1),\ C(kT_1),\ D(kT_1)) \ , \text{ if } i = p_1 - 1 \end{cases} . \tag{3.2}$$

If some of the inputs to the discrete-time subsystem have this property and others do not, the proper τ-varying representation is a hybrid of (3.1) and (3.2). The justification for this modification is that when $T_2/T_1 \in \mathbb{N}$, the inputs to the sample/hold of period T_1 or the T_1-sampled inputs of a discrete-time subsystem are held constant over the times

$$t = kT_1,\ kT_1 + \tau, \ldots,\ kT_1 + (p_1-1)\tau;$$

$u(kT_1)$ is available, if necessary, at times after $t = kT_1$ for determining outputs and state transitions.

A slight modification of Procedure 3.1 produces a τ/q-varying representation, $q \in \mathbb{N}$. By introducing a fictitious sampling period $T_0 = \tau/q$ into the normalization process in Procedure 3.1(b), a different set of parameters results: $\tau' = \tau/q$, $P' = qP$, and $p_i' = qp_i$. Utilizing these parameters in place of τ, P, and p_i in steps (c), (d), and (e) of Procedure 3.1 then gives a τ'-varying representation.

3.2 The $m\tau$-varying Representation

Obtaining an $m\tau$-varying representation, $m \in \mathbb{N}$, for a multirate system by direct means can be somewhat tedious. However, a method will be outlined whereby an $m\tau$-varying representation can be obtained via the τ-varying representation in a straightforward manner. In the following procedure, assume that the system is single-bundle input, single-bundle output.

Procedure 3.2:

a. Apply Procedure 3.1 to obtain $(A(n\tau),B(n\tau),C(n\tau),D(n\tau))$.

b. Let $r = \mathrm{LCM}\,\{m,P\}/m$. For $k = 0,1,\ldots,$ calculate

$$\alpha(km\tau) = A((km+m-1)\tau)\ldots A((km+1)\tau)A(km\tau),$$

$$\beta(km\tau) = [A((km+m-1)\tau)\ldots A((km+1)\tau)B(km\tau) \mid \ldots$$
$$\ldots \mid A((km+m-1)\tau)B((km+m-2)\tau) \mid B((km+m-1)\tau)],$$

$$\gamma(km\tau) = \begin{bmatrix} C(km\tau) \\ C((km+1)\tau)A(km\tau) \\ \vdots \\ C((km+m-1)\tau)A((km+m-2)\tau)\ldots A(km\tau) \end{bmatrix},$$

and the block $m \times m$ matrix $\delta(km\tau)$, where

$$\delta_{ij}(km\tau) = \begin{cases} 0 \text{ , if } i < j \\ D((km+i-1)\tau) \text{ , if } i = j \\ C((km+i-1)\tau)B((km+j-1)\tau) \text{ , if } i = j+1 \\ C((km+i-1)\tau)A((km+i-2)\tau)\ldots A((km+j)\tau)B((km+j-1)\tau) \text{ , if } i > j+1 \end{cases}$$

Since $\alpha(.)$, $\beta(.)$, $\gamma(.)$, and $\delta(.)$ are periodic with period $rm\tau$, only values for $k = 0,\ldots, r-1$ are required. With $z(n\tau)$, $u(n\tau)$, and $y(n\tau)$ as the state, input, and output of the τ-varying representation, the $m\tau$-varying representation satisfies the following state and output equation for $k = 0,1,\ldots$:

$$z((k+1)m\tau) = \alpha(km\tau)\, z(km\tau) + \beta(km\tau)\, U(km\tau)$$
$$Y(km\tau) = \gamma(km\tau)\, z(km\tau) + \delta(km\tau)\, U(km\tau).$$

$Y(km\tau)$ and $U(km\tau)$ are the $m\tau$-expanded versions of $u(n\tau)$ and $y(n\tau)$.

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Iteration of the equations governing the τ-varying representation confirms the validity of Procedure 3.2. Note the similarity between these formulas and those for the time expansion of a periodic discrete-time system in Section 2.7. $Y(km\tau)$ and $U(km\tau)$ in Procedure 3.2 should not be confused with the $m\tau$-expanded versions of the output and input of the original system. $Y(km\tau)$ and $U(km\tau)$ are $m\tau$-expanded versions of τ-sampled versions of the original system's input and output. In fact, the representation produced by Procedure 3.2 is the $m\tau$-varying representation of the multirate system with samplers of period τ inserted at each input and output of the original system. These added samplers do not alter the behavior of the system. Since the inputs and outputs of a multirate system are generally sampled with periods greater than τ, many of the components of $U(km\tau)$ are not accessed by the $m\tau$-varying representation and many of the components of $Y(km\tau)$ are duplicates. This redundancy serves to pad the inputs and outputs of the $m\tau$-varying representation to a size which is uniform in time.

Example 3.2:

Consider finding the 3τ-varying representation of the multirate system in Example 3.1. Since $r = 6/3 = 2$, only two values for each of α, β, γ, and δ are needed. The τ-varying representation of this system was determined in Example 3.1; Procedure 3.2(a) is finished. By straightforward calculations,

$k = 0$:

$$\alpha(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \beta(0) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \gamma(0) = \begin{bmatrix} c_1 & 0 & 0 \\ c_1 & 0 & 0 \\ c_2 & 0 & 0 \end{bmatrix}, \delta(0) = \begin{bmatrix} d_1 & 0 & 0 \\ d_1 & 0 & 0 \\ d_2 & 0 & 0 \end{bmatrix}.$$

$$Y(km\tau) = \begin{bmatrix} y(0) \\ y(\tau) \\ y(2\tau) \end{bmatrix}, \text{ and } U(km\tau) = \begin{bmatrix} u(0) \\ u(\tau) \\ u(2\tau) \end{bmatrix}.$$

k = 1:

$$\alpha(3\tau) = \begin{bmatrix} a & b_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \beta(3\tau) = \begin{bmatrix} b_2 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \gamma(3\tau) = \begin{bmatrix} c_2 & d_2 & 0 \\ c_3 & d_3 & 0 \\ c_3 & d_3 & 0 \end{bmatrix}, \delta(3\tau) = \begin{bmatrix} 0 & 0 & 0 \\ d_4 & 0 & 0 \\ d_4 & 0 & 0 \end{bmatrix},$$

$$Y(km\tau) = \begin{bmatrix} y(3\tau) \\ y(4\tau) \\ y(5\tau) \end{bmatrix}, \text{ and } U(km\tau) = \begin{bmatrix} u(3\tau) \\ u(4\tau) \\ u(5\tau) \end{bmatrix}.$$

Thus,

$$(\alpha,\beta,\gamma,\delta)\ (k(3\tau)) = \begin{cases} (\alpha,\beta,\gamma,\delta)\ (0)\ , \text{ if } k = 0,2,4,\ldots \\ (\alpha,\beta,\gamma,\delta)\ (3\tau)\ , \text{ if } k = 1,3,5,\ldots \end{cases}.$$

Since y has period 2τ, it should be the case that $y(0) = y(\tau)$, $y(2\tau) = y(3\tau)$, and $y(4\tau) = y(5\tau)$, which can be verified. Note that only $u(0)$ and $u(3\tau)$ are utilized, which is consistent with the fact that u is updated with period 3τ by the original multirate system.

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Procedure 3.2 determines the $m\tau$-varying representation of a single-bundle input, single-bundle output, multirate system. The $m\tau$-varying representation of a multirate system with inputs and outputs which are vectors of bundles results from partitioning $B(n\tau)$, $C(n\tau)$, and $D(n\tau)$ and repeated application of Procedure 3.2(b) in a manner analogous to the time expansion of such multirate systems. As noted earlier, τ/q-varying representations, $q \in \mathbb{N}$, may be obtained using Procedure 3.1. It follows from the developments in this section that if $r \in \mathbb{Q}$ and $r > 0$, Procedure 3.2 may be used to calculate the $r\tau$-varying representation of a multirate system, since $r = m/q$ for some integers m and q.

3.3 Corresponding Representations

A trait common to all periodic representations of multirate systems, but not the T-expanded representation, is that states corresponding to the contents of some of the hold circuits in the system must be included in the composite state vector of the system. These added states are of a

nondynamic nature, and the reader may correctly predict that the lower-dimensional T-expanded representation of a given multirate system shares many of the qualitative properties of a periodic representation of that system. The representations described in this section serve as a crucial step in the investigation in Chapter 4 of the manner in which the T-expanded representation relates to periodic representations. The properties and results discussed in this section and Chapter 4 regarding T-expanded and τ-varying representations could be adapted to almost any periodic representation of the types of multirate systems under consideration.

For a given multirate system, a relationship exists between the matrices $A(n\tau)$ of the τ-varying representation obtained from Procedure 3.1 and the matrix A_e of the T-expanded representation obtained from Procedure 2.4, provided that two particular steps of these procedures are performed similarly. To ensure that such representations exist, let the following assumption be satisfied.

Assumption 3.1: (well-posedness)

The calculations necessary to complete Procedure 2.4(i) and Procedure 3.1(e) can be performed for the multirate systems under consideration and involve matrices with bounded elements.

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The state of the T-expanded representation, $x(kT)$, is a composite of the states of all the dynamic subsystems in the multirate system. The state of the τ-varying representation, $z(n\tau)$, is a composite of the states of all the dynamic subsystems in the system and hold states (states assigned to sample/hold circuits and states introduced to retain input values of discrete-time subsystems.) A T-expanded and a τ-varying representation of a multirate system will be called **corresponding representations** if these representations are obtained from Procedure 2.4 and Procedure 3.1, respectively, and the following convention is satisfied.

Convention 3.3:

a. The same system partition and the same state space realization for each subsystem is used in step (c) of Procedures 2.4 and 3.1.

b. When reducing the composite system to a single state space equation in Procedure 3.1(e), $z(n\tau)$ is partitioned into dynamic states, $w(n\tau)$, and hold states, $h(n\tau)$:

$$z(n\tau) = \begin{bmatrix} w(n\tau) \\ h(n\tau) \end{bmatrix}.$$

In addition, Procedure 3.1(e) and Procedure 2.4(i) are performed so that the state of any given subsystem occupies the same components of x and w.

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If corresponding representations have initial conditions z(0) and x(0), where x(0) = w(0), and are subjected to the same input, then x(kT) = w(kPτ), k = 0,1,..., because x and w each represent the dynamic states of the system and are structured identically. Hereafter, this fact is emphasized by writing

$$z(n\tau) = \begin{bmatrix} x(n\tau) \\ h(n\tau) \end{bmatrix},$$

where x(nτ) is interpreted through Convention 3.1 as representing the dynamic states of the subsystems in the multirate system at time t = nτ.

Let x be $\delta \times 1$ and h be $\eta \times 1$. As a consequence of the manner in which τ-varying representations were defined in Section 3.1, at time t = 0 each subsystem and sample/hold determines its next state and current output solely from the current value of its dynamic states (if any) and inputs. Thus, at t = 0 the next state z(τ) is independent of every component of h(0). If the inputs to the system are zero, from z(τ) = A(0)z(0), it follows that for some $E_1 \in \mathbb{R}^{\delta\times\delta}$ and $E_2 \in \mathbb{R}^{\eta\times\delta}$,

$$A(0) = \begin{bmatrix} E_1 & 0 \\ E_2 & 0 \end{bmatrix}.$$

Expressing z(Pτ) = z(T) for arbitrary z(0) as

$$z(p\tau) = A((P-1)\tau)...A(\tau)A(0)z(0) = \begin{bmatrix} G & 0 \\ H & 0 \end{bmatrix} z(0)$$

for some $G \in \mathbb{R}^{\delta\times\delta}$ and $H \in \mathbb{R}^{\eta\times\delta}$, it follows that x(Pτ) = Gx(0) for arbitrary x(0). For the corresponding T-expanded representation, $x(T) = A_e x(0)$ and x(kT) = x(kPτ). Thus,

$$A_e x(0) = x(T) = x(P\tau) = Gx(0)$$

for arbitrary x(0), which implies that $G = A_e$. The following theorem results from the discussion above.

Theorem 3.1:

For corresponding T-expanded and τ-varying representations of a multirate system and some $E_1 \in \mathbb{R}^{\delta\times\delta}$ and $E_2, H \in \mathbb{R}^{\eta\times\delta}$,

$$A(0) = \begin{bmatrix} E_1 & 0 \\ E_2 & 0 \end{bmatrix} \text{ and } A((P-1)\tau)...A(\tau)A(0) = \begin{bmatrix} A_e & 0 \\ H & 0 \end{bmatrix}.$$

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A T-expanded and a τ-varying representation of a multirate system will be called **completely corresponding representations** if these representations are corresponding representations and the following convention is satisfied.

Convention 3.4:

a. Samplers of period τ are placed on each input and output of the block diagram of the system before finding the T-expanded representation.

b. Once the T-expanded representation has been obtained, its input and output, U(kT) and Y(kT), are restructured to form a new input and output, $U(kP\tau)$ and $Y(kP\tau)$, given by

$$U(kP\tau) = \begin{bmatrix} u(kP\tau) \\ u(kP\tau+\tau) \\ \vdots \\ u((k+1)P\tau-\tau) \end{bmatrix} \text{ and } Y(kP\tau) = \begin{bmatrix} y(kP\tau) \\ y(kP\tau+\tau) \\ \vdots \\ y((k+1)P\tau-\tau) \end{bmatrix}.$$

In essence, the input and output of the system are each treated as a single bundle of signals. During this operation, the rows, columns, and elements of the matrices of the representation must be permuted according to the reordering of the inputs and outputs.

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Let the completely corresponding T-expanded representation be denoted by

$$x((k+1)T) = A_e\, x(kT) + B_{cc}\, U(kP\tau)$$
$$Y(kP\tau) = C_{cc}\, x(kT) + D_{cc}\, U(kP\tau).$$

The primary difference between a corresponding and a completely corresponding T-expanded representation is Convention 3.4(a); however, the added samplers do not alter the behavior of the system. In fact, the columns of B_{cc} are permutations of the columns of B_e interspersed with columns of zeroes, and the rows of C_{cc} contain duplicates and are drawn from the rows of C_e. For a given multirate system, a τ-varying representation which is a corresponding representation and one which is a completely corresponding representation are identical.

The manner in which completely corresponding representations are related is revealed by defining the following quantities from the matrices $(A(n\tau),B(n\tau),C(n\tau),D(n\tau))$ of the τ-varying representation:

$$\alpha = A((P-1)\tau)\ldots A(\tau)A(0),$$

$$\beta = [A((P-1)\tau)\ldots A(\tau)B(0) \mid \ldots \mid A((P-1)\tau)B((P-2)\tau) \mid B((P-1)\tau)],$$

$$\gamma = \begin{bmatrix} C(0) \\ C(\tau)A(0) \\ \vdots \\ C((P-1)\tau)A((P-2)\tau)\ldots A(0) \end{bmatrix},$$

and the block $P \times P$ matrix δ, where

$$\delta_{ij} = \begin{cases} 0 \text{ , if } i < j \\ D((i-1)\tau) \text{ , if } i = j \\ C((i-1)\tau)B((j-1)\tau) \text{ , if } i = j+1 \\ C((i-1)\tau)A((i-2)\tau)\ldots A(j\tau)B((j-1)\tau) \text{ , if } i > j+1 \end{cases}$$

Note the similarity of these quantities to the formulas in Procedure 3.2 for a $P\tau$-varying representation of a multirate system whose inputs and outputs are treated as a single bundle. It follows that

$$z((k+1)P\tau) = \alpha\, z(kP\tau) + \beta\, U(kP\tau)$$
$$Y(kP\tau) = \gamma\, z(kP\tau) + \delta\, U(kP\tau).$$

For completely corresponding representations with dynamic states which are identical at $t = 0$ and input $U(kP\tau)$, not only will $x(kT) = x(kP\tau)$, as with corresponding representations, but in addition their outputs $Y(kP\tau)$ will be identical. In particular, with $z(0) = 0$, $x(0) = 0$, and $U(0)$ arbitrary,

$$x(T) = B_{cc}\, U(0), \quad Y(0) = D_{cc}\, U(0)$$

and

$$z(P\tau) = \begin{bmatrix} x(P\tau) \\ h(P\tau) \end{bmatrix} = \beta\, U(0), \quad Y(0) = \delta\, U(0).$$

Since $U(0)$ is arbitrary and $x(T) = x(P\tau)$, β must be of the form

$$\beta = \begin{bmatrix} B_{cc} \\ J \end{bmatrix}$$

for some matrix J and $\delta = D_{cc}$. By the manner in which τ-varying representations were defined in Section 3.1, at time $t = 0$ each subsystem and sample/hold determines its next state and current output solely from the current value of its dynamic states (if any) and inputs. Thus, for some matrices E_1, E_2, and E_3,

$$A(0) = \begin{bmatrix} E_1 & 0 \\ E_2 & 0 \end{bmatrix} \text{ and } C(0) = [E_3\ 0].$$

By definition of γ, it follows that for some matrix E_4, $\gamma = [E_4\ 0]$. Consider setting $U(0) = 0$ and letting $z(0)$ and $x(0)$ be arbitrary. Then

$$\gamma\, z(0) = [E_4\ 0]\begin{bmatrix} x(0) \\ h(0) \end{bmatrix} = Y(0) = C_{cc}\, x(0);$$

therefore, $\gamma = [C_{cc}\ 0]$. The following theorem summarizes the above discussion.

Theorem 3.2:

For completely corresponding T-expanded and τ-varying representations of a multirate system and some matrices H and J,

$$\alpha = \begin{bmatrix} A_e & 0 \\ H & 0 \end{bmatrix}, \ \beta = \begin{bmatrix} B_{cc} \\ J \end{bmatrix}, \ \gamma = [C_{cc} \ 0], \ \text{and } \delta = D_{cc},$$

where α, β, and γ are partitioned conformal with the partition of z into dynamic and hold states.

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CHAPTER 4

ANALYSIS OF MULTIRATE SYSTEMS

The representations developed in the previous two chapters provide a convenient means of assessing important qualitative characteristics of multirate systems. Of particular value is the fact that the stability, controllability, reconstructibility, stabilizability, and detectability of the τ-varying representation may be determined by examining the corresponding T-expanded representation of the system.

In this chapter, only systems that satisfy Assumption 3.1 (well-posedness) are considered. Throughout, denote the τ-varying representation of the multirate system by

$$z((n+1)\tau) = A(n\tau)\, z(n\tau) + B(n\tau)\, u(n\tau)$$
$$y(n\tau) = C(n\tau)\, z(n\tau) + D(n\tau)\, u(n\tau),$$

where $(A,B,C,D)\,(n\tau)$ is T-periodic, and

$$z(n\tau) = \begin{bmatrix} x(n\tau) \\ h(n\tau) \end{bmatrix} \in \Re^{\delta+\eta} .$$

Denote the corresponding T-expanded representation of this system (see Section 3.3 for details) by

$$x((k+1)T) = A_e\, x(kT) + B_e\, U(kT)$$
$$Y(kT) = C_e\, x(kT) + D_e\, U(kT)$$

and the completely corresponding T-expanded representation of this system by

$$x((k+1)T) = A_e\, x(kT) + B_{cc}\, U(kP\tau)$$
$$Y(kP\tau) = C_{cc}\, x(kT) + D_{cc}\, U(kP\tau),$$

where $x \in \Re^{\delta}$ and $T = P\tau$. By Assumption 3.1, the elements of $(A,B,C,D)\,(n\tau)$, (A_e,B_e,C_e,D_e), and $(A_e,B_{cc},C_{cc},D_{cc})$ are bounded and well-defined.

4.1 Stability

Clearly, a necessary condition for uniform asymptotic stability of a multirate system is that its T-expanded representation be asymptotically stable, i.e., all the eigenvalues of A_e have

magnitudes strictly less than one. It turns out that this is also a sufficient condition for uniform asymptotic stability of the corresponding τ-varying representation.

Theorem 4.1:

If the eigenvalues of A_e have magnitudes less than 1, then the τ-varying representation is uniformly asymptotically stable.

Proof:

Let $Z(n,n_0,z_0)$ be the zero input state trajectory at time $t = n\tau$ of the τ-varying representation resulting from the initial state z_0 at time $t = n_0\tau$. It suffices to show that $\forall\ n_0 \in \mathbb{N}$ and $z_0 \in \mathbb{R}^{\delta+\eta}$, $\lim_{n\to\infty} Z(n,n_0,z_0) = 0$, as this result and the periodicity of the τ-varying representation establish uniformity.

Assume the eigenvalues of A_e have magnitudes less than 1. Define the monodromy matrix at time $n\tau$ for the τ-varying representation as

$$\Phi(n) = A((n+P-1)\tau)...A((n+1)\tau)A(n\tau).$$

Note that $\Phi(n+kP) = \Phi(n)$ for $k = 0,1,...$ and that under zero input

$$z((n+P)\tau) = \Phi(n)z(n\tau) \text{ for } n = 0,1,... .$$

By Theorem 3.1, for some $H \in \mathbb{R}^{\eta\times\delta}$,

$$\Phi(0) = \begin{bmatrix} A_e & 0 \\ H & 0 \end{bmatrix}.$$

Thus, each eigenvalue of $\Phi(0)$ has magnitude less than one. Recall that for square matrices V and W, VW and WV have the same eigenvalues. Repeated application of this fact leads to the conclusion that $\Phi(0)$, $\Phi(1)$,..., and $\Phi(P-1)$ all have identical eigenvalues. Let $i \in \{1,...,P\}$ and $k = 0,1,...$ and note that

$$Z(kP+i+n_0,n_0,z_0) = [\Phi(n_0+i)]^k A((n_{0+i-1})\tau)...A(n_0\tau)z_0.$$

For any $z_0 \in \mathbb{R}^{\delta+\eta}$ and any $n_0 \in \mathbb{N}$, $A((n_{0+i-1})\tau)...A(n_0\tau)z_0$ is bounded for each $i \in \{1,...,P\}$. Since the eigenvalues of each $\Phi(n)$ have magnitudes less than 1, for each $i \in \{1,...,P\}$ $Z(kP+i+n_0,n_0,z_0)$ is bounded and $\lim_{k\to\infty} Z(kP+i+n_0,n_0,z_0) = 0$; $Z(n,n_0,z_0)$ is partitioned into P bounded convergent subsequences. Thus, $Z(n,n_0,z_0)$ is bounded and for any $i \in \{1,...,P\}$ and $\varepsilon > 0$, there is $K(i,\varepsilon) \in \mathbb{N}$ such that $\| Z(kP+i+n_0,n_0,z_0) \| < \varepsilon\ \forall\ k \geq K(i,\varepsilon)$. It follows that for any $\varepsilon > 0$ there is $K(\varepsilon) = 1 + \max\{K(i,\varepsilon): i \in \{1,...,P\}\}$ such that $\| Z(n,n_0,z_0) \| < \varepsilon$ for all $n \geq K(\varepsilon)P + n_0$, which implies that $\lim_{n\to\infty} Z(n,n_0,z_0) = 0$.

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The utility of Theorem 4.1 stems from the fact that T-expanded representations are generally easier to calculate and are of lower order than τ-varying representations. Theorem 4.1 also insures that $x(n\tau)$, rather than merely $x(kT)$, decays asymptotically to zero for any initial condition if the T-expanded representation is asymptotically stable. In fact, consider collecting the states of all continuous-time subsystems in the multirate system into the composite state $x_c(t)$. For $k = 0,1,...$ and $0 \le t_0 < P\tau$, write

$$x_c\,(kP\tau+t_0) = \Psi(t_0)\; z(kP\tau), \tag{4.1}$$

where $\Psi(t_0)$ is a $\dim(x_c) \times (\delta+\eta)$ matrix. Under the assumption that $\|\Psi(t_0)\|$ is bounded for each $t_0 \in [0,P\tau)$, if the eigenvalues of A_e have magnitude less than 1, it follows from (4.1) and Theorem 4.1 that as $t \to \infty$, $x_c(t) \to 0$. As a consequence of Theorem 4.1 and the satisfaction of Assumption 3.1 by the systems under consideration, if the eigenvalues of A_e have magnitudes less than 1, the τ-varying and T-expanded representations are BIBO stable (Chen, 1987).

4.2 Controllability and Reconstructibility

The controllability, reachability, reconstructibility, and observability of linear periodic discrete-time systems have been investigated in several recent works; see Grasselli (1984), Bittani and Bolzern (1985), Bittani and Colaneri (1986), Bittani and Guardabassi (1986) and the references cited therein. The τ-varying representation provides a T-periodic description of a multirate system amenable to analysis by the methods developed in these references. Such an analysis may be quite involved because the properties of controllability, reachability, etc. of a linear periodic system are in general enjoyed by only a time-varying subspace of the entire state space. The analysis here presents two instances in which the τ-varying representation inherits a property of its corresponding T-expanded representation.

The informal definitions which follow are consistent with those of Grasselli (1984). A linear, discrete-time system is said to be **controllable** if there exist inputs which drive the system from any given initial state at any given time to the zero state in a finite amount of time. A linear, discrete-time system is said to be **reachable** if there exist inputs which drive the system to any given terminal state at any given point in time from the zero state in a finite amount of time. A linear, T_0-varying discrete-time system is said to be **reconstructible** if the state of the system at any time $t = mT_0$ can be determined from knowledge of the inputs and outputs of the system over some finite interval of time prior to $t = mT_0$. A linear, T_0-varying discrete-time system is said to be **observable** if the state of the system at any time $t = mT_0$ can be determined from knowledge of the inputs and outputs of the system over some finite interval of time beginning at $t = mT_0$.

Determination of the controllability, reachability, reconstructibility, or observability of the T-expanded representation can be accomplished by considering (A_e,B_e,C_e) as a time-invariant triple. A consequence of the Cayley-Hamilton theorem is that at most δ T-expanded inputs are required to drive the state $x(kT)$ to zero if the T-expanded representation is controllable or to the desired terminal state if the T-expanded representation is reachable. In addition, at most δ T-expanded outputs are required to determine a present state $x(kT)$ of the T-expanded representation if it is reconstructible or to determine an initial state $x(kT)$ of the T-expanded representation if it is observable.

Theorem 4.2:

The τ-varying representation is controllable if the corresponding T-expanded representation is controllable, and any initial condition $z(n_0\tau) = z_0 \in \mathbb{R}^{\eta+\delta}$ can be driven to the zero state by time $t = n_0\tau + (\delta+1)T$.

Proof:

Let $n_0 = k_0P - i$, $i \in \{0,1,...,P-1\}$. If $i > 0$ and the inputs at times $n_0\tau,..., (n_0+i-1)\tau$ are zero, the system progresses from z_0 to some state

$$z(k_0P\tau) = \begin{bmatrix} x(k_0P\tau) \\ h(k_0P\tau) \end{bmatrix},$$

where $x(k_0P\tau) = x(k_0T) \in \mathbb{R}^{\delta}$. Since the T-expanded representation is controllable, *there is an input sequence* which drives $x(k_0T)$ to the origin by time $t = k_0T + \delta T$ at the latest. If $x(kT)$ reaches the origin before $t = k_0T + \delta T$, zero inputs can be applied to ensure that $x(kT)$ remains at the origin until this time. Applying this input sequence to the τ-varying representation,

$$z((k_0+\delta)P\tau) = \begin{bmatrix} 0 \\ h((k_0+\delta)P\tau) \end{bmatrix}.$$

From Theorem 3.1, with $u((k_0+\delta)P\tau) = 0$,

$$z((k_0+\delta)P\tau+\tau) = \begin{bmatrix} E_1 & 0 \\ E_2 & 0 \end{bmatrix} z((k_0+\delta)P\tau) = 0.$$

Thus, an arbitrary state z_0 has been driven to 0 by time

$$t = (n_0+i+1+\delta P)\tau \leq n_0\tau + (\delta+1)T.$$

♦♦♦

Theorem 4.3:

The τ-varying representation is reconstructible if the corresponding T-expanded representation is reconstructible, and the state of the corresponding representation at any time $n_0\tau$, $z(n_0\tau)$, may be determined by examining the inputs

and outputs of the system over a period of time no greater than $(\delta+1)T$ prior to $t = n_0\tau$.

Proof:

Let $n_0 = k_0P + j$, $j \in \{1,...,P\}$. Since the T-expanded representation is reconstructible, the state $x(k_0T) = x(k_0P\tau)$ may be determined by examining the input $U(kT)$ and output $Y(kT)$ of the T-expanded representation at the times $(k_0-\delta)T,...,(k_0-1)T$. From Theorem 3.1,

$$z((k_0P+1)\tau) = \begin{bmatrix} E_1 & 0 \\ E_2 & 0 \end{bmatrix}\begin{bmatrix} x(k_0P\tau) \\ h(k_0P\tau) \end{bmatrix} + B(0)\,u(k_0P\tau) = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} x(k_0T) + B(0)\,u(k_0P\tau).$$

Thus, $z((k_0P+1)\tau)$ may be determined from knowledge of $x(k_0T)$. It follows that $z((k_0P+j)\tau) = z(n_0\tau)$ may be determined for any $j \in \{1,...P\}$ by examining the inputs and outputs of the system at times $(k_0-\delta)P\tau,...,(k_0P+i-1)\tau$ (recall that $Y(kT)$ involves values of y at and after time $kP\tau$.) Therefore, the τ-varying representation is reconstructible and its present state can be determined by examining inputs and outputs of the system over a period of time $(n_0\tau - (k_0-\delta)P\tau) \le (\delta+1)T$ prior to $t = n_0\tau$.

♦♦♦

Besides the application to determining the controllability or reconstructibility of the τ-varying representation of a multirate system, Theorems 4.2 and 4.3 contain a subtler result. Results in Grasselli (1984) show that a τ-varying, T-periodic, discrete-time linear system of order $\delta + \eta$, such as the τ-varying representation under consideration, can be driven from $z(n_0\tau)$ to the zero state by at most time $t = n_0\tau + (\delta + \eta)T$ if it is controllable and that the state of the system at time $t = n_0\tau$ can be determined from knowledge of its inputs and outputs over a period of time no greater than $(\delta + \eta)T$ prior to $t = n_0\tau$ if it is reconstructible. From the special properties of τ-varying representations and their corresponding T-expanded representations, Theorems 4.2 and 4.3 shorten these time bounds by $(\eta-1)T$.

4.3 Duality, Stabilizability, and Detectability

In analogy with time-invariant discrete-time state space representations, the reader may anticipate the existence of stabilizability and detectability properties and dual relationships between stabilizability and detectability for periodic discrete-time representations. By combining results from Grasselli (1984), Grasselli and Lampariello (1981), and Weiss (1972), this is seen to be the case. In the remainder of this section, the argument τ will be suppressed.

The approach taken in defining stabilizability and detectability will be through a decomposition based on controllability and reconstructibility. The following lemma is a partial restatement of results in Grasselli (1984).

Lemma 4.1:

A T-periodic coordinate transformation $\tilde{z}(n) = \Lambda(n)z(n)$, $\Lambda(n)$ nonsingular for each n, exists such that if $\tilde{\Sigma}(n) = (\tilde{A}(n), \tilde{B}(n), \tilde{C}(n), \tilde{D}(n))$ is the representation of $\Sigma(n)$ in the new coordinates,

$$\tilde{\Sigma}(n) = (\Lambda(n+1)A(n)\Lambda^{-1}(n), \Lambda(n+1)B(n), C(n)\Lambda^{-1}(n), D(n)),$$

then

$$\tilde{A}(n) = \begin{bmatrix} \tilde{a}_{11}(n) & \tilde{a}_{12}(n) & \tilde{a}_{13}(n) & \tilde{a}_{14}(n) \\ 0 & \tilde{a}_{22}(n) & 0 & \tilde{a}_{24}(n) \\ 0 & 0 & \tilde{a}_{33}(n) & \tilde{a}_{34}(n) \\ 0 & 0 & 0 & \tilde{a}_{44}(n) \end{bmatrix}, \quad \tilde{B}(n) = \begin{bmatrix} \tilde{b}_1(n) \\ \tilde{b}_2(n) \\ 0 \\ 0 \end{bmatrix},$$

$$\tilde{C}(n) = [\, 0 \;\; \tilde{c}_2(n) \;\; 0 \;\; \tilde{c}_4(n) \,], \text{ and } \tilde{D}(n) = D(n). \tag{4.1}$$

All of the submatrices appearing in (4.1) are T-periodic and have dimensions which are <u>constant</u> with n. In addition, $\tilde{a}_{11}(n)$, $\tilde{a}_{33}(n)$, and $\tilde{a}_{44}(n)$ are square and nonsingular for all n,

$$\left(\begin{bmatrix} \tilde{a}_{11}(n) & \tilde{a}_{12}(n) \\ 0 & \tilde{a}_{22}(n) \end{bmatrix}, \begin{bmatrix} \tilde{b}_1(n) \\ \tilde{b}_2(n) \end{bmatrix}, [\, 0 \;\; \tilde{c}_2(n) \,], \tilde{D}(n) \right)$$

is controllable, and

$$\left(\begin{bmatrix} \tilde{a}_{22}(n) & \tilde{a}_{24}(n) \\ 0 & \tilde{a}_{44}(n) \end{bmatrix}, \begin{bmatrix} \tilde{b}_2(n) \\ 0 \end{bmatrix}, [\, \tilde{c}_2(n) \;\; \tilde{c}_4(n) \,], \tilde{D}(n) \right)$$

is reconstructible.

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As a consequence of the results stated in Lemma 4.1, the subsystems $(\tilde{a}_{11}(n), \tilde{b}_1(n), 0, \tilde{D}(n))$, $(\tilde{a}_{22}(n), \tilde{b}_2(n), \tilde{c}_2(n), \tilde{D}(n))$, $(\tilde{a}_{33}(n), 0, 0, \tilde{D}(n))$, and $(\tilde{a}_{44}(n), 0, \tilde{c}_4(n), \tilde{D}(n))$ are, respectively, controllable and unreconstructible, controllable and reconstructible, uncontrollable and unreconstructible, and uncontrollable and reconstructible. By the nonsingularity of $\tilde{a}_{11}(n)$, $\tilde{a}_{33}(n)$, and $\tilde{a}_{44}(n)$, any zero eigenvalues of $[\tilde{A}(P-1)\ldots\tilde{A}(1)\tilde{A}(0)]$ must appear in the controllable and reconstructible subsystem. In light of Theorem 3.1, the controllable and reconstructible portion of the τ-varying representation has dimension of at least η. This property may serve to thwart attempts to lower the dimension of the representation by discarding all subsystems except the controllable and reconstructible subsystem; the remaining system will have dimension of at least η.

It is suspected that a decomposition based on reachability and observability would avoid this problem; however, the dimensions of subsystems in such a decomposition in general vary with time.

Based on the decomposition in Lemma 4.1, stabilizability and detectability can be defined.

Definition 4.1:

$\Sigma(n)$ will be said to be **stabilizable** if its uncontrollable portion is asymptotically stable; with reference to the decomposition in Lemma 4.1, the eigenvalues of $[\tilde{a}_{33}(P-1)...\tilde{a}_{33}(1)\,\tilde{a}_{33}(0)]$ and of $[\tilde{a}_{44}(P-1)...\tilde{a}_{44}(1)\,\tilde{a}_{44}(0)]$ all have magnitudes less than 1.

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Definition 4.2:

$\Sigma(n)$ will be said to be **detectable** if its unreconstructible portion is asymptotically stable; with reference to the decomposition in Lemma 4.1, the eigenvalues of $[\tilde{a}_{33}(P-1)...\tilde{a}_{33}(1)\,\tilde{a}_{33}(0)]$ and of $[\tilde{a}_{11}(P-1)...\tilde{a}_{11}(1)\,\tilde{a}_{11}(0)]$ all have magnitudes less than 1.

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The following lemma, which arises from results in Weiss (1972) and the periodicity of $\Sigma(n)$, was noted in Grasselli and Lampariello (1981).

Lemma 4.2:

$\Sigma(n) = (A(n),B(n),C(n),D(n))$ is controllable (reconstructible) if and only if the dual of $\Sigma(n)$, $\Sigma_d(n) = (A'(-n),B'(-n),C'(-n),D'(-n))$, is reconstructible (controllable).

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By making use of the decomposition in Lemma 4.1 and the duality properties in Lemma 4.2, stabilizability and detectability can be shown to be dual properties.

Theorem 4.4:

$\Sigma(n)$ is stabilizable (detectable) if and only if $\Sigma_d(n)$ is detectable (stabilizable.)

Proof:

Let the dual representation have state $\zeta(n)$ and let

$$\Sigma_d(n) = (A'(-n),\, B'(-n),\, C'(-n),\, D'(-n)) = (\alpha(n),\, \beta(n),\, \gamma(n),\, \delta(n)).$$

Consider the coordinate transformation $\tilde{\zeta}(n) = \Theta(n)\zeta(n)$, where $\Theta(n) = [\Lambda^{-1}(1-n)]'$. The representation of $\Sigma_d(n)$ in these coordinates, $\tilde{\Sigma}_d(n)$, is given by

$$\tilde{\alpha}(n) = \Theta(n+1)\,\alpha(n)\,\Theta^{-1}(n) = [\Lambda^{-1}(-n)]'\,A'(-n)\,[\Lambda(1-n)]' = [\tilde{A}(-n)]',$$

$$\tilde{\beta}(n) = \Theta(n+1)\,\beta(n) = [\Lambda^{-1}(-n)]'\,C'(-n) = [\tilde{C}(-n)]',$$

and similarly, $\tilde{\gamma}(n) = [\tilde{B}(-n)]'$ and $\tilde{\delta}(n) = [\tilde{D}(-n)]'$. Thus,

$$\tilde{\alpha}(n) = \begin{bmatrix} \tilde{a}'_{11}(-n) & 0 & 0 & 0 \\ \tilde{a}'_{12}(-n) & \tilde{a}'_{22}(-n) & 0 & 0 \\ \tilde{a}'_{13}(-n) & 0 & \tilde{a}'_{33}(-n) & 0 \\ \tilde{a}'_{14}(-n) & \tilde{a}'_{24}(-n) & \tilde{a}'_{34}(-n) & \tilde{a}'_{44}(-n) \end{bmatrix}, \quad \tilde{\beta}(n) = \begin{bmatrix} 0 \\ \tilde{c}'_2(-n) \\ 0 \\ \tilde{c}'_4(-n) \end{bmatrix},$$

$$\tilde{\gamma}(n) = [\, \tilde{b}'_1(-n) \;\; \tilde{b}'_2(-n) \;\; 0 \;\; 0 \,], \text{ and } \tilde{\delta}(n) = \tilde{D}'(-n). \tag{4.2}$$

From the properties of the submatrices in (4.2) given in Lemma 4.1 and from Lemma 4.2, it can be seen that $\tilde{a}'_{33}(-n)$ and $\tilde{a}'_{44}(-n)$ determine the dynamics of the unreconstructible portion of $\Sigma_d(n)$ and $\tilde{a}'_{11}(-n)$ and $\tilde{a}'_{33}(-n)$ determine the dynamics of the uncontrollable portion of $\Sigma_d(n)$. Note that

$$\begin{aligned} \tilde{a}'_{11}(-(P-1))\ldots\tilde{a}'_{11}(-1)\;\tilde{a}'_{11}(0) &= \tilde{a}'_{11}(1)\ldots\tilde{a}'_{11}(P-1)\;\tilde{a}'_{11}(0) \\ &= [\,\tilde{a}_{11}(0)\,\tilde{a}_{11}(P-1)\ldots\tilde{a}_{11}(1)]'. \end{aligned}$$

Since $\tilde{a}'_{11}(.)$ is square, the eigenvalues of $\tilde{a}'_{11}(-(P-1))\ldots\tilde{a}'_{11}(-1)\tilde{a}'_{11}(0)$ are equal to the eigenvalues of $\tilde{a}_{11}(P-1)\ldots\tilde{a}_{11}(1)\,\tilde{a}_{11}(0)$. A similar relation holds for $\tilde{a}'_{33}(-n)$ and $\tilde{a}'_{44}(-n)$. Thus, the uncontrollable (unreconstructible) portion of $\Sigma(n)$ is *asymptotically stable* if and only if the unreconstructible (uncontrollable) portion of $\Sigma_d(n)$ is asymptotically stable. The statement of the theorem then follows from Definitions 4.1 and 4.2.

♦♦♦

The examination of the relationship between corresponding T-expanded and τ-varying representations can now be completed.

Lemma 4.3:

(A(n), B(n), C(n), D(n)) is stabilizable if there exist T-periodic matrices F(n) such that on applying the state feedback

$$u(kP+i) = -F(kP+i)\; z(kP), \quad 0 \le i < P, \quad k = 0,1,\ldots,$$

the resulting representation is asymptotically stable; that is, the closed-loop monodromy matrix at time t = 0,

$$\Phi_c(0) = A(P-1)\,[\ldots[A(1)\,[A(0) - B(0)\,F(0)] - B(1)\,F(1)]\ldots] - B(P-1)\,F(P-1),$$

has eigenvalues of magnitude less than 1.

Proof:

Without loss of generality, it can be assumed that (A(n), B(n), C(n), D(n)) has been transformed into the canonical form in Lemma 4.1; it can be verified that if

state feedback $\tilde{F}(n) = F(n)\,\Lambda(0)$ is used in the $\tilde{z}$ coordinates, then

$$\tilde{\Phi}_c(0) = \Lambda^{-1}(P)\,\Phi_c(0)\,\Lambda(0) = \Lambda^{-1}(0)\,\Phi_c(0)\,\Lambda(0).$$

Thus, $\tilde{\Phi}_c(0)$ and $\Phi_c(0)$ have the same eigenvalues. Partition $(\tilde{A}(n), \tilde{B}(n))$ into its controllable and uncontrollable parts:

$$\tilde{A}(n) = \begin{bmatrix} \tilde{A}_{11}(n) & \tilde{A}_{12}(n) \\ 0 & \tilde{A}_{22}(n) \end{bmatrix}, \quad \tilde{B}(n) = \begin{bmatrix} \tilde{B}_1(n) \\ 0 \end{bmatrix}.$$

Note that if there is $\tilde{F}(n)$ such that each eigenvalue of

$$\tilde{\Phi}_c(0) = \tilde{A}(P-1)\,[\ldots[\tilde{A}(1)\,[\tilde{A}(0) - \tilde{B}(0)\,\tilde{F}(0)] - \tilde{B}(1)\,\tilde{F}(1)]\ldots] - \tilde{B}(P-1)\,\tilde{F}(P-1)$$

has magnitude less than 1, then each eigenvalue of $[\tilde{A}_{22}(P-1)\ldots\tilde{A}_{22}(1)\,\tilde{A}_{22}(0)]$ has magnitude less than 1. This is easily seen by expanding $\tilde{\Phi}_c(0)$ as

$$\tilde{\Phi}_c(0) = \tilde{A}(P-1)\ldots\tilde{A}(1)\,\tilde{A}(0) - \sum_{j=0}^{P-2}\,[\tilde{A}(P-1)\ldots\tilde{A}(j+1)\tilde{B}(j)\tilde{F}(j)] - \tilde{B}(P-1)\,\tilde{F}(P-1)$$

and observing that $\tilde{A}(P-1)\ldots\tilde{A}(j+1)$ is upper triangular for $-1 \le j \le P-2$. From the special form of $\tilde{B}(n)$, $\tilde{\Phi}_c(0)$ is upper triangular with $[\tilde{A}_{22}(P-1)\ldots\tilde{A}_{22}(1)\,\tilde{A}_{22}(0)]$ appearing on the diagonal. Thus, if the eigenvalues of $\tilde{\Phi}_c(0)$ have magnitudes less than 1, so do the eigenvalues of $[\tilde{A}_{22}(P-1)\ldots\tilde{A}_{22}(1)\,\tilde{A}_{22}(0)]$, and $(A(n), B(n))$ is stabilizable.

♦♦♦

Theorem 4.5:

The τ-varying representation of a multirate system is stabilizable if its corresponding T-expanded representation is stabilizable.

Proof:

The theorem can be restated equivalently as "the τ-varying representation of a multirate system is stabilizable if its completely corresponding T-expanded representation is stabilizable," since the input matrix of the completely corresponding representation, B_{cc}, is merely a rearrangement of the columns of the input matrix of the corresponding representation, B_e, padded with columns of zeroes.

If the completely corresponding T-expanded representation is stabilizable, it is a standard time-invariant result that there is F_{cc} such that the eigenvalues of $(A_e - B_{cc}F_{cc})$ have magnitudes less than 1. Consider applying state feedback to the τ-varying representation as in Lemma 4.3:

$$u(kP+i) = -F(kP+i)\,z(kP), \quad 0 \le i < P, \quad k = 0,1,\ldots .$$

Suppose $u \in \Re^{\rho \times 1}$. Partition F_{cc} into P groups of rows,

$$F_{cc} = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{P-1} \end{bmatrix},$$

where $f_i \in \mathbb{R}^{\rho \times \delta}$. Set $F(n) = [\ f_n \ \ 0_{\rho \times \eta}]$ and write $\Phi_c(0)$ as

$$\Phi_c(0) = A(P\text{-}1)...A(1)\ A(0) - \sum_{j=0}^{P-2} [A(P\text{-}1)...A(j+1)B(j)F(j)] \ - B(P\text{-}1)\ F(P\text{-}1).$$

In the notation of Theorem 3.2,

$$\Phi_c(0) = \alpha - \beta \begin{bmatrix} F(0) \\ F(1) \\ \vdots \\ F(P\text{-}1) \end{bmatrix}.$$

From the values of F(n) and Theorem 3.2,

$$\Phi_c(0) = \begin{bmatrix} A_e & 0 \\ H & 0 \end{bmatrix} - \begin{bmatrix} B_{cc} \\ J \end{bmatrix} [\ F_{cc} \ \ 0_{P\rho \times \eta}].$$

Thus, the eigenvalues of $\Phi_c(0)$ have magnitudes less than 1, and by Lemma 4.3, the τ-varying representation is stabilizable.

♦♦♦

By using Theorem 3.2, Theorem 4.4, and Lemma 4.3 and proceeding in a manner parallel to Theorem 4.5, the following theorem can also be established.

Theorem 4.6:

The τ-varying representation of a multirate system is detectable if its corresponding T-expanded representation is detectable.

♦♦♦

The statements of Theorems 4.2, 4.3, 4.5, and 4.6 all hold with "if" replaced by "only if," but the new assertion is rather unimportant and in general trivial to prove. By arguments parallel to those presented in this chapter, it can be shown that M-varying representations which correspond to T-expanded representations (in a manner analogous to corresponding τ-varying and T-expanded representations) inherit the stability, controllability, reconstructibility, stabilizability, and detectability properties of the T-expanded representation.

CHAPTER 5

DESIGN OF MULTIRATE CONTROLLERS

The representations and methods of analysis developed for multirate systems thus far bear fruit in the form of multirate controllers. The term "multirate controller" need not imply that only the controller is multirate; candidates for control include all systems to which Procedure 2.4 and its extensions apply. Although the T-expanded or M-varying representations are not truly time invariant or periodic due to the time expanded form of their inputs and outputs, this characteristic offers little or no obstruction to the design and implementation of multirate controllers by conventional means. The controller designs presented here are intended as illustrations of the general properties of controller design and implementation using T-expanded and M-varying representations of multirate systems. The usefulness of these representations is by no means limited to the controllers examined here.

5.1 T-expanded State Feedback and Observers

Consider a multirate system with the T-expanded representation

$$\begin{aligned} x((k+1)T) &= A_e\, x(kT) + B_{ve}\, V(kT) + B_{re}\, R(kT) \\ Y(kT) &= C_e\, x(kT) + D_{ve}\, V(kT) + D_{re}\, R(kT), \end{aligned} \tag{5.1}$$

where $V(kT)$ is an input intended for control purposes and $R(kT)$ is a reference or load input. On applying feedback of the form $V(kT) = -F\, x(kT)$, the state equation becomes

$$x((k+1)T) = [A_e - B_{ve}F]\, x(kT) + B_{re}R(kT).$$

If a matrix F can be found such that the eigenvalues of $[A_e - B_{ve}F]$ are all less than 1 in magnitude, then Theorem 4.1 asserts that this representation, as well as the corresponding τ-varying representation, is asymptotically stable. Conditions for the existence of such a matrix F given (A_e, B_{ve}) are well known, and methods of calculating F abound.

Although calculation of F by treating (A_e, B_{ve}) as a time-invariant pair using pole placement or linear quadratic methods serves the purpose of stabilizing the system, two comments are in order regarding the choice of design parameters for these methods. When choosing pole

locations for pole placement, bear in mind that these pole locations are in a time scale of T seconds per transition. An LQ design with even weighting on each member of an expanded bundle of inputs with period T_1, say $V_1(kT)$, may result in a control law which consistently produces inputs $v_1(kT)$ of magnitude quite different from $v_1((k+1)T-T_1)$. The cause of this behavior is that if the subsystems affected by v_1 have poles far from the origin (close to the origin,) then an input $v_1(kT)$ will have a greater (lesser) effect on $x((k+1)T)$ than an input $v_1((k+1)T-T_1)$. If such behavior is deemed undesirable, it can be predicted by examining the columns of B_{ve} corresponding to $V_1(kT)$ and corrected by appropriate changes in the input weighting matrix.

The design of observers which produce an estimate $\hat{x}(kT)$ of $x(kT)$ parallels the design of such observers in the standard time-invariant case, with two minor exceptions. The observer must produce a predictive estimate, usually denoted by $\hat{x}(kT \mid (k-1)T)$, since current estimates, commonly denoted by $\hat{x}(kT \mid kT)$, result in a noncausal observer. The observer must also account for the D_{ve} and D_{re} terms frequently present in the T-expanded representation.

The state equation for an observer for system (5.1) in predictive form is

$$\hat{x}((k+1)T) = A_e\hat{x}(kT) + B_{ve}V(kT) + B_{re}R(kT) + K[Y(kT) - C_e\hat{x}(kT) - D_{ve}V(kT) - D_{re}R(kT)].$$

The estimation error satisfies $x((k+1)T) - \hat{x}((k+1)T) = [A_e - KC_e](x(kT) - \hat{x}(kT))$. If K is chosen to stabilize $[A_e - KC_e]$ by methods dual to those employed to find F, this error will asymptotically approach zero as $k \to \infty$. The state and output equations for the combined state feedback and observer pair are

$$\begin{aligned} \hat{x}((k+1)T) &= [A_e - KC_e - [B_{ve} - KD_{ve}]F]\,\hat{x}(kT) + KY(kT) + [B_{re} - KD_{re}]R(kT) \\ V(kT) &= -F\,\hat{x}(kT). \end{aligned} \tag{5.2}$$

It is easily verified that the principle of separation holds for the state feedback and observer designs. Equation (5.2) is a multirate discrete-time system specified in T-expanded form with input $[Y(kT)'\ R(kT)']'$ and output $V(kT)$. As $\hat{x}((k+1)T)$ is needed at time $t = (k+1)T$ to compute $V((k+1)T)$ and each component of $Y(kT)$ and $R(kT)$ is available at time $t = ((k+1)T - \tau)$ at the latest, it follows that the controller given above is causal. Although all of the information necessary to calculate $\hat{x}((k+1)T)$ may not be present until time $t = ((k+1)T - \tau)$, the calculation of

$$[A_e - KC_e - [B_{ve} - KD_{ve}]F]\,\hat{x}(kT)$$

is possible immediately after $t = kT$. From

$$KY(kT) = K_{1,1}y_1(kT) + K_{1,2}y_1(kT+T_1) + \ldots + K_{2,1}y_2(kT) + K_{2,2}y_2(kT+T_2) + \ldots,$$

where the $K_{i,j}$ are groups of columns of K chosen to be compatible with the individual values of the i^{th} bundle of signals (with period T_i) comprising $Y(kT)$ at time $t = kT + jT_i$, the calculation of $KY(kT)$ and $[B_{re} - KD_{re}]R(kT)$ can occur progressively as the values of $Y(kT)$ and $R(kT)$ become available. The implementation of (5.2) involves sampling each component of $y(t)$ and $r(t)$ at the

appropriate rate and transferring values of -F $\hat{x}(kT)$ to each component of $v(t)$ at the appropriate rate and holding each of these values until the next arrives. This behavior is conceptualized as sample/hold devices at appropriate rates at the inputs and outputs of the controller. The following procedure summarizes the steps necessary to find a **T-expanded multirate state feedback-observer controller.**

Procedure 5.1:

a. Decide upon the rate desired for each of the controller outputs and the desired rate at which each plant output and reference input is to be observed by the controller. Place samplers of corresponding periods on the block diagram of the system after the controller output v and the reference input r and before the controller input y, as shown in Figure 5.1. Assume that only the components of r observed by the controller are fed to the summer, as in Figure 5.1. Relaxation of this assumption is considered after this procedure.

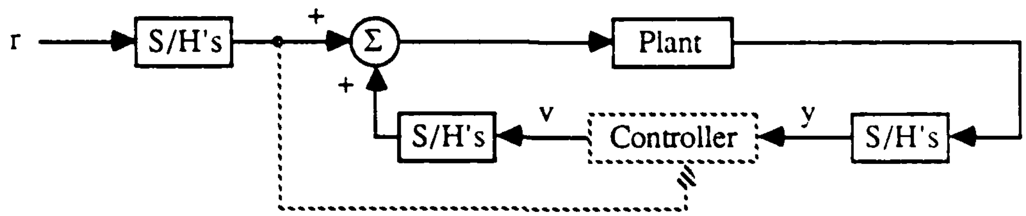

Figure 5.1. Placement of samplers.

b. Apply Procedure 2.4 and any modifications which apply to find a T-expanded representation such as (5.1), treating r and v as inputs and y as the output.

c. Using any method desired, find matrices F and K from the pairs (A_e, B_{ve}) and (A_e, C_e) such that each eigenvalue of $[A_e - B_{ve}F]$ and $[A_e - KC_e]$ has magnitude less than 1. The controller state and output equations are given by (5.2).

♦♦♦

The assumption in Procedure 5.1(a) that only the components of the reference input observed by the controller are fed to the summer is necessary for the design procedure but need not be satisfied in the implementation of the controller. Let r_s be the actual sampled and held version of r applied to the plant, where each component of r_s has a period which is an integer multiple of τ, the fundamental period used in the execution of Procedure 5.1. Let r_o be the sampled version of r observed by the controller. Since the components of r_s and r_o vary at times which are integer multiples of τ, r_s can be written $r_s = r_o + r_u$, where the components of r_u each vary with period τ.

The signal r_u is the portion of r_s unobserved by the controller. Let the plant and controller system with r_o and r_u as inputs have composite state

$$s(kT) = \begin{bmatrix} x(kT) \\ \tilde{x}(kT) \end{bmatrix}$$

and state equation

$$s((k+1)T) = A\, s(kT) + B_o\, R_o(kT) + B_u\, R_u(kT). \tag{5.3}$$

A is stable by design; therefore, under the quite reasonable assumption that B_o and B_u are bounded, (5.3) is bounded-input bounded-state stable. If it is desired that the controller sample r at a slow rate, then either r must be fed to the plant at this slow rate, slowing the system's response time, or r can be fed to the plant at a fast rate, in which case the unobserved portion r_u acts as a disturbance.

The T-expanded multirate controller design has the advantages of ease of calculation by established methods and being readily implemented. To its disadvantage, this controller exhibits a rather sluggish response to rapid changes in r if T is large. The state feedback $V(kT) = -F\,\tilde{x}(kT)$ is essentially applied open loop over times of length T, leaving the system vulnerable to disturbances and inputs r_u unobserved by the controller. Indeed, if the observer portion is initialized to $\tilde{x}(0) = 0$, no control will be applied until time $t = T$. The design of a T-expanded multirate Kalman filter for a system with noise at the input which is uncorrelated with the noise at the output is complicated by the fact that the D_e term in the T-expanded representation introduces noise at the output which is highly correlated with the noise at the input.

5.2 M-varying State Feedback and Observers

With the intent of achieving a faster response to inputs and disturbances, the value of the state may be estimated and fed back at times $t = n\tau$. Consider the τ-varying representation of the multirate system in Figure 5.1 (in the following, the parameter τ is suppressed):

$$\begin{aligned} z(n+1) &= A(n)\, z(n) + B_v(n)\, v(n) + B_r(n)\, r(n) \\ y(n) &= C(n)\, z(n) + D_v(n)\, v(n) + D_r(n)\, r(n). \end{aligned} \tag{5.4}$$

After applying state feedback $v(n) = -F(n)\, z(n)$,

$$z(n+1) = [A(n) - B_v(n)F(n)]\, z(n) + B_r(n)\, r(n).$$

If $F(0), \dots, F(P-1)$ can be found such that each eigenvalue of the closed-loop monodromy matrix at $t = 0$,

$$\Phi_c(0) = [A(P-1) - B_v(P-1)F(P-1)] \dots [A(1) - B_v(1)F(1)][A(0) - B_v(0)F(0)],$$

has magnitude less than 1, then the system employing $F(n) = F(n \bmod P)$ as T-periodic feedback matrices will be asymptotically stable. Such a periodic feedback arises naturally as the solution to the LQ problem with the cost function

$$J = 1/2 \sum_{n=0}^{\infty} [z'(n)W_z'(n)W_z(n)z(n) + v'(n)W_v'(n)W_v(n)v(n)], \tag{5.5}$$

where $W_z(n+P) = W_z(n)$ and $W_v(n+P) = W_v(n)$ for each $n \in \mathbb{Z}^+$. The discrete-time Riccati equation associated with this problem admits a T-periodic solution which determines the feedback matrices F(n). Discrete-time periodic Riccati equations have just recently been investigated; conditions for the existence and uniqueness of their solutions, methods of calculating these solutions, and conditions under which the resulting feedback is stabilizing are still developing. See Bittani et al. (1986) and the references cited therein.

In order to make use of the results in Bittani et al. (1986), assume that $W_v(n)$ is square and nonsingular for each $n \in \mathbb{Z}^+$. The Riccati equation associated with (5.4) and (5.5) can then be written

$$\begin{aligned} S(n) &= A'(n)S(n+1)A(n) + W_z'(n)W_z(n) \\ &\quad - A'(n)S(n+1)B_w(n)\,[I + B_w'(n)S(n+1)B_w(n)]^{-1}\,B_w'(n)S(n+1)A(n), \end{aligned} \tag{5.6}$$

where $B_w(n) = B_v(n)(W_v(n))^{-1}$. For state feedback $v(n) = -F(n)\,z(n)$,

$$\begin{aligned} F(n) &= (W_v(n))^{-1}\,[I + B_w'(n)S(n+1)B_w(n)]^{-1}\,B_w'(n)S(n+1)A(n) \\ &= [W_v'(n)W_v(n) + B_v'(n)S(n+1)B_v(n)]^{-1}\,B_v'(n)S(n+1)A(n). \end{aligned} \tag{5.7}$$

The following theorem is stated without proof; its proof relies on the application of results in Bittani et al. (1986), the duality properties of discrete-time periodic systems (see Theorem 4.4) and the assumed nonsingularity of $W_v(n)$.

Theorem 5.1:

Consider the dynamical system represented by

$$\begin{aligned} z(n+1) &= A(n)\,z(n) + B_v(n)\,v(n) \\ \tilde{y}(n) &= W_z(n)\,z(n), \end{aligned} \tag{5.8}$$

where $A(n)$ and $B_v(n)$ are as in (5.4) and $W_z(n)$ is as in (5.5). Then a unique T-periodic symmetric positive semidefinite solution to (5.6) exists and (5.4) is asymptotically stable after applying feedback $v(n) = -F(n)\,z(n)$, where F(n) is given by (5.7), if and only if (5.8) is stabilizable and detectable; i.e., the uncontrollable part of (5.8) is asymptotically stable and the unreconstructible part of (5.8) is asymptotically stable.

♦♦♦

By Theorem 4.5, the stabilizability of (5.8) can be evaluated by examining the stabilizability of the

T-expanded representation (with $r(n) \equiv 0$) corresponding to (5.4). This device cannot be employed to determine whether (5.8) is detectable; however, if $W_z(n)$ is square and nonsingular for each n, (5.8) is detectable.

A τ-varying observer to form an estimate $\hat{z}(n)$ of $z(n)$ takes a form identical to the T-expanded observer:

$$\hat{z}(n+1) = A(n)\hat{z}(n) + B_v(n)v(n) + B_r(n)r(n) + K(n)[y(n) - C(n)\hat{z}(n) - D_v(n)v(n) - D_r(n)r(n)].$$

Since $z(n+1) - \hat{z}(n+1) = [A(n) - K(n)C(n)]\,(z(n) - \hat{z}(n))$, the error $z(n) - \hat{z}(n)$ will asymptotically approach zero if $K(n)$ is chosen in a manner dual to the choice of $F(n)$ for stabilization (see Theorem 4.4 for the required duality details.) By writing the equations of the combined controller and plant in terms of $z(n)$ and $(z(n) - \hat{z}(n))$ and finding the T-expanded representation of this τ-varying representation, it can be verified that the principle of separation holds for the state feedback and observer designs.

The procedure for finding a τ-varying, T-periodic state feedback-observer controller via the discrete-time periodic Riccati equation parallels Procedure 5.1. Comments similar to those following Procedure 5.1 with regard to reference inputs unobserved by the controller also apply.

Using the periodic description (5.4) for the multirate system, other types of τ-varying controllers can be designed. Let $z(n) \in \Re^{\delta+\eta}$, as in Chapter 4. Results in Grasselli and Lampariello (1981) show that the reconstructibility and controllability of (5.4) (with $r(n) \equiv 0$) are necessary and sufficient for the existence of a τ-varying, T-periodic state feedback-observer dead-beat controller for (5.4) which is capable of driving any initial condition to the origin within an interval of time no greater than $2(\delta+\eta)T$. In light of Theorems 4.2 and 4.3, it is suspected that a closer examination of the dead-beat controller problem could reduce this interval to $2(\delta+1)T$ for the representation (5.4). M-varying controllers can be found by utilizing the M-varying representation of the multirate system in Figure 5.1. As M-varying representations involve M-expanded inputs and outputs, care must be taken to insure that the controller designed is causal. If the M-varying controller is of the state feedback-observer type, the controller will be causal if the observer produces predictive estimates.

Implementation of a τ-varying controller requires an interpretation of $v(n)$, $r(n)$, and $y(n)$. In Procedure 5.1(a), sample/hold devices are placed on the block diagram to represent the effects of the sampling and "output and hold" operations performed by the discrete controller. The fact that values of v, r, and y are specified for the controller at times $t = n\tau$ might suggest that the τ-varying controller be implemented by loading values of r, y, and v into and out of the controller at times $t = n\tau$ and physically passing these values through the sample/hold devices following v and r and preceding y on the block diagram of Procedure 5.1(a). The implementation can be made much more efficient by bringing the sample/hold devices inside the controller in the following manner. For each component $v_i(n)$ of $v(n)$, if n corresponds to a time at which the sample/hold

associated with v_i samples, then $v_i(n)$ should be output by the controller at time $t = n\tau$ and held for the duration of the sampling period associated with v_i. For each component $y_j(n)$ of $y(n)$, if n corresponds to a time at which the sample/hold associated with y_j samples, then $y_j(n)$ should be acquired by the controller at time $t = n\tau$ and stored for the duration of the sampling period associated with y_j to provide the values $y_j(n)$, $y_j(n+1)$,... . The signals $r(n)$ are treated similar to $y(n)$. Analogous comments hold for M-varying controllers.

A τ-varying controller can potentially respond quickly to changes in the input r and, since the state estimate is updated frequently, reduce the effects of unmeasured disturbances on the output. The price paid for these attributes takes the form of complexity of design and implementation; the τ-varying representation employed by the controller has more states than the T-expanded representation, resulting in greater computation and storage requirements. An M-varying controller, where M is chosen such that $T/M \in \mathbb{N}$, may serve as an adequate compromise between τ-varying and T-expanded controllers; by choosing different values of M, complexity can be traded for response time. The practicality of solving the discrete-time periodic Riccati equation or finding the feedback matrices required in Grasselli and Lampariello (1981) is unknown as of the present.

5.3 Transfer Function Controller Designs

By thinking of (5.1) as a time-invariant state and output equation, a transfer function matrix may be obtained for this system:

$$Y(z) = \left(C_e(zI - A_e)^{-1} [B_{ve}\ B_{re}] + [D_{ve}\ D_{re}] \right) \begin{bmatrix} V(z) \\ R(z) \end{bmatrix} = G(z) \begin{bmatrix} V(z) \\ R(z) \end{bmatrix}.$$

Using MIMO transfer function design techniques, controllers of a variety of structures may be designed (Kailath, 1980). One such structure is

$$V(z) = V_y(z) + V_r(z) = H_y(z)\, Y(z) + H_r(z)\, R(z).$$

Although details of such a design will not be elaborated upon here, the restrictions imposed on $H_y(z)$ and $H_r(z)$ by the T-expanded nature of $V(kT)$, $Y(kT)$, and $R(kT)$ warrant discussion. A state space realization of $H_y(z)$, if it exists, takes the form of a multirate discrete-time system specified in T-expanded form,

$$\begin{aligned} w((k+1)T) &= A_y\, w(kT) + B_y\, Y(kT) \\ V_y(kT) &= C_y\, w(kT) + D_y\, Y(kT), \end{aligned} \tag{5.9}$$

where $D_y = H_y(\infty)$. For this system to be physically realizeable, the elements of $H_y(\infty)$ must be finite; $H_y(z)$ must be proper. For (5.9) to be causal, selected entries in D_y must be zero, and the

corresponding entries in $H_y(z)$ must be strictly proper. Let $H_y(z)$ be partitioned into $i \times j$ blocks $h_{ij}(z)$ to conform with the signal bundle values $v(kT+t_i)$ comprising $V(kT)$ and $y(kT+t_j)$ comprising $Y(kT)$ (this notation is expedient and unrelated to previous notations.) For $H_y(z)$ to be realizable as a causal multirate system, it must satisfy

$$h_{ij}(\infty) = \begin{cases} 0 \text{ , if } t_i < t_j \\ \text{a matrix of finite numbers , if } t_j \leq t_i \end{cases} . \tag{5.10}$$

Multirate controllers designed by transfer function techniques possess some of the advantages of both the T-varying and τ-varying state space controllers. The T-expanded nature of such a controller simplifies its implementation, and the possibility of incorporating "D" terms into the state space realization of the controller may allow the controller output during [kT, (k+1)T) to depend on its inputs during [kT, (k+1)T), avoiding the open loop type behavior of the T-expanded state space controller with respect to measured disturbances. However, the state of a multirate controller designed by transfer function techniques is updated only at times t = kT, which may impair its ability to reduce the effects of unmeasured disturbances on the output. The practical implications of incorporating the causality constraints (5.10) into MIMO transfer function design techniques are unknown at the present.

5.4 Comments

The advantages of using a multirate controller as opposed to a single-rate controller depend greatly on the specific system to be controlled and on an intelligent choice of sampling periods for the controller. Clearly, if the plant is inherently multirate, conventional methods of analysis and design are not applicable to controller design. However, multirate systems frequently arise from attempts to circumvent the "a controller is only as fast as its slowest actuator or sensor" principle of single-rate sampled-data designs. In such a situation, the simplicity of a single-rate controller must be weighed against the prospects of making full use of the bandwidth of the actuators and sensors by multirate control. Results in Barnes and Shinnaka (1980) indicate that implementations of multirate systems specified in T-expanded form exhibit desirable numerical characteristics, such as low roundoff noise. A basic property of multirate systems is that as the numbers p_i obtained from the normalization process increase, representation and analysis of the system become more difficult. Exploiting any freedom in the choice of actuator and sensor rates to reduce the numbers p_i will result in a much simpler multirate controller.

CHAPTER 6

THE VARIABLE COMPONENT METHOD

APPLIED TO MULTIRATE DISCRETE-TIME SYSTEMS

The variable component method and the method of sensitivity points have been applied to synthesize and tune linear, time-invariant controllers for linear, time-invariant plants; see Frank (1978), Kokotovic (1964), Kokotovic (1965), Hung (1985), and the references cited therein. In this chapter, the variable component method will be extended to the class of multirate discrete-time systems to which Procedure 2.4 and its extensions apply. A general time-invariant variable component result will be derived and then related to the T-expanded representation of these multirate systems.

Consider a time-invariant discrete-time single-rate system with an embedded scalar parameter k and scalar output y(u(n),k,n) at time n for a given scalar input u(n). The variable component method and the method of sensitivity points are each based on a block diagram representation of the system and provide a means of determining the **output sensitivity function**, $\partial[y(u(n),k,n)]/\partial k$, by simulation or implementation. These two methods basically proceed by injecting the input u(n) into the system or a suitable model of the system and extracting selected signals, as determined by the block diagram, from the system or model and injecting these signals (or possibly filtered versions of these signals) at selected points of another model of the system. Alternately, the extracted signals may be stored and injected into the actual system at a later time. In either case, the response at a selected point of the two connected systems is the output sensitivity function. In addition to the actual system, the variable component method requires as many models of the system as there are parameters to be independently varied if the output sensitivity function for each parameter is to be obtained simultaneously. The method of sensitivity points allows the output sensitivity functions for any number of parameters to be obtained simultaneously with only one system–model pair. However, the method of sensitivity points cannot be applied to MIMO systems (Hung, 1985).

Once the output sensitivity function has been obtained, the response of the system with parameter value $k + \Delta k$, $|\Delta k| \ll |k|$, and input u(n) can be approximated by

$$y(u(n),k+\Delta k,n) \approx y(u(n),k,n) + \Delta k(\partial[y(u(n),k,n)]/\partial k).$$

The output sensitivity function thus provides information useful for iteratively adjusting the parameter k so that the output of the system meets or approaches certain time-domain criteria for a given input u(n). Such adjustments are frequently determined through the solution of an optimization problem by minimizing a measure of the error between the actual output of the system and the desired output of the system (see Kokotovic (1965) and Hung (1985).) An example of the use of the variable component method with a multirate system would be the tuning of parameters in a single-rate sampled-data controller based on intersample values of the output of the controlled system.

6.1 A Time-invariant Result

In the following, a general variable component method result for time-invariant discrete-time MIMO systems will be derived. This time-invariant result can then be applied to the T-expanded representation of a multirate discrete-time system.

Consider a MIMO time-invariant single-rate discrete-time linear system with rate 1/T and a parameter $K = \mathrm{diag}[k_1,k_2,\ldots,k_r]$ which can be isolated as shown in Figure 6.1. Let U(z), E(z), and Y(z) be the vector Z-transforms of U(mT), E(mT), and Y(mT), respectively, and let U(z) be $j \times 1$, E(z) be $r \times 1$, and Y(z) be $q \times 1$. The dependence of E(z) and Y(z) on K and U(z) will not be made explicit at this point.

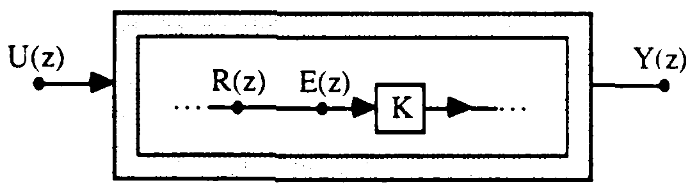

Figure 6.1. A MIMO time-invariant system.

Assuming that the system in Figure 6.1 is well-posed (see Chen (1984),) there exist transfer function matrices F(z), G(z), H(z), and J(z) which do not depend on the parameters $k_1, k_2, \ldots, k_r$ such that, suppressing the argument z,

$$R = GU + HKE = GU + HKR.$$

Thus,

$$R = (I_r - HK)^{-1}\, GU. \tag{6.1}$$

Also,

$$Y = FU + JKR = FU + JK\,(I_r - HK)^{-1}\, GU. \tag{6.2}$$

For the set $S \subseteq \{1,2,...,r\}$, define the $r \times r$ matrix $\Delta(r,S)$ by

$$\Delta_{ij}(r,S) = \begin{cases} 1 \ , \text{ if } i = j \in S \\ 0 \ , \text{ otherwise} \end{cases}.$$

Note that $\partial K/\partial k_i = \Delta(r,\{i\})$. From (6.2), we can compute

$$\begin{aligned} \partial Y/\partial k_i &= J\,\Delta(r,\{i\})\,(I_r - HK)^{-1}\,GU + JK\,(I_r - HK)^{-1}\,H\,\Delta(r,\{i\})\,(I_r - HK)^{-1}\,GU \\ &= J\,[I_r + (I_r - KH)^{-1}\,KH]\,\Delta(r,\{i\})\,(I_r - HK)^{-1}\,GU \\ &= J\,(I_r - KH)^{-1}\,[I_r - KH + KH]\,\Delta(r,\{i\})\,(I_r - HK)^{-1}\,GU \\ &= J\,(I_r - KH)^{-1}\,\Delta(r,\{i\})\,(I_r - HK)^{-1}\,GU. \end{aligned} \tag{6.3}$$

Since $(\partial Y/\partial k_i)k_i = \partial Y/\partial \ln|k_i|$ and $K\,\Delta(r,\{i\}) = k_i\,\Delta(r,\{i\})$, it follows from (6.3) that

$$\partial Y/\partial \ln|k_i| = J\,(I_r - KH)^{-1}\,K\,\Delta(r,\{i\})\,(I_r - HK)^{-1}\,GU. \tag{6.4}$$

Consider taking a copy of the system in Figure 6.1 with zero inputs, inserting a summer into it, and connecting it to the system in Figure 6.1 as shown in Figure 6.2.

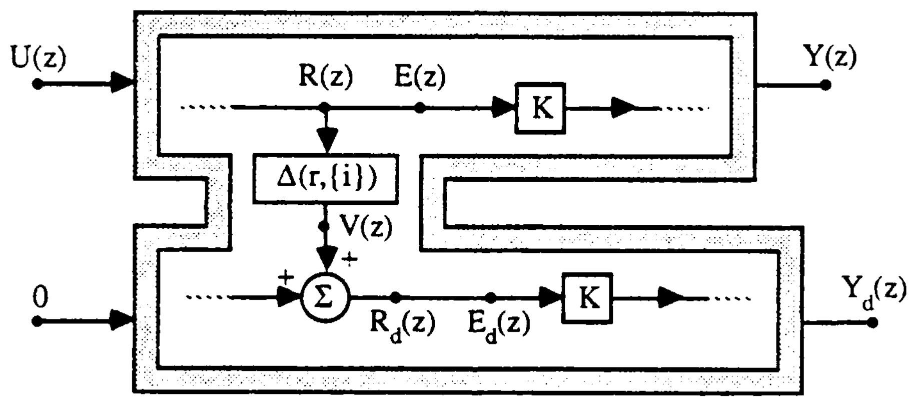

Figure 6.2. The system in Figure 6.1 connected to a duplicate of itself.

Then $R_d = HKE_d + V = HKR_d + V$; hence, $R_d = (I_r - HK)^{-1}\,V$ and

$$Y_d = JKR_d = JK\,(I_r - HK)^{-1}\,V = J\,(I_r - KH)^{-1}\,KV.$$

From $V(z) = \Delta(r,\{i\})\,R(z)$ and (6.1),

$$Y_d = J\,(I_r - KH)^{-1}\,K\,\Delta(r,\{i\})\,R = J\,(I_r - KH)^{-1}\,K\,\Delta(r,\{i\})\,(I_r - HK)^{-1}\,GU. \tag{6.5}$$

Comparing (6.5) with (6.4),

$$Y_d(z) = \partial Y(z)/\partial \ln|k_i|. \tag{6.6}$$

Since the operations of Z-transforming and differentiation with respect to a parameter can be interchanged, $Y_d(mT) = \partial Y(mT)/\partial \ln|k_i|$. As additional notation, let $Y_d(z,S)$ be the output of the

duplicate system in Figure 6.2 when $V(z) = \Delta(r,S)\, R(z)$. In this notation, $Y_d(z)$ in Figure 6.2 is $Y_d(z,\{i\})$. By linearity,

$$Y_d(mT,S) = \sum_{i\in S} \partial Y(mT)/\partial \ln|k_i|. \tag{6.7}$$

It may be desired to set $k_i = k$ for $i \in S \subseteq \{1,2,\ldots,r\}$ and determine $\partial Y(mT)/\partial \ln|k|$ to examine the effect of changing the parameters k_i, $k \in S$, in unison. Treating each k_i as a function of k,

$$k_i(k) = \begin{cases} k\ , & i \in S \\ c_i\ , & i \notin S \end{cases},$$

where c_i are constants independent of k, and using the chain rule for differentiation,

$$\begin{aligned}\partial Y(mT)/\partial \ln|k| &= \sum_{i=1}^{r} [(\partial Y(mT)/\partial \ln|k_i|)\,(\partial \ln|k_i|/\partial \ln|k|)] \\ &= \sum_{i\in S} \partial Y(mT)/\partial \ln|k_i|.\end{aligned}$$

From (6.7),

$$\partial Y(mT)/\partial \ln|k| = Y_d(mT,S). \tag{6.8}$$

In particular, if $K = \mathrm{diag}[k,k,\ldots,k]$ then $S = \{1,2,\ldots,r\}$ and $\partial Y(mT)/\partial \ln|k|$ is found by setting $V(mT) \equiv R(mT)$. By a similar argument, it follows that if the parameter k appears at several locations in the original system, $\partial Y(mT)/\partial \ln|k|$ will be the output of the duplicate system if the signal entering the gain block k is extracted from the original system at each occurrence of k and summed with the signal entering the gain block k at each corresponding occurrence of k in the duplicate system. Note that if k_1 and k_2 are two parameters which are to be varied independently, the original system and two copies of it are required to determine $\partial Y(mT)/\partial \ln|k_1|$ and $\partial Y(mT)/\partial \ln|k_2|$ simultaneously.

The quantity $\partial Y(mT)/\partial \ln|k_i|$ is referred to as a **semirelative output sensitivity function** in the literature (Frank, 1978). With reference to Figure 6.2, the **output sensitivity function**, $\partial Y(mT)/\partial |k_i|$, may be obtained by summing V(mT) with the output of the gain block K in the duplicate system.

6.2 Interpretation of the Time-invariant Result

Consider a multirate discrete-time system with input u and output y which contains a discrete-time subsystem that is a SISO periodic gain $g(nT_1)$ with input u_s, output y_s, and period $T_2 = qT_1$. Thus, this gain sweeps cyclically through a set of values $\{g_1,\ldots,g_q\}$ and the

subsystem can be described by

$$y_s(nT_1) = g(nT_1)\, u_s(nT_1) = g_{((n \bmod q)+1)}\, u_s(nT_1).$$

Let $T/T_2 = s_2$, where T is the period of the entire system and $s_2 \in \mathbb{N}$. From (2.6), the T-expanded representation of this subsystem is $Y_s(mT) = K\, U_s(mT)$, where with $G = \mathrm{diag}[g_1,g_2,...,g_q]$, $K = \mathrm{diag}[G,G,...,G]$; G appears s_2 times along the diagonal of K.

In light of the time-invariant nature of T-expanded representations and the fact that the structure of the block diagram of a T-expanded representation is identical to that of the block diagram of the original representation, it is seen that the T-expanded representation of the system under consideration here is amenable to the application of the variable component method as described in the preceding section. This entails passing the signals entering the gain K in the block diagram of the T-expanded representation of the system through a matrix gain and summing the output of this matrix gain with the input of the gain K in a duplicate of the T-expanded representation of the system. To be consistent with the preceding section, let $r = qs_2$. The r parameters composing K cannot be varied independently since each parameter g_i appears at s_2 places in K. Let K be represented as $K = \mathrm{diag}[k_1,k_2,...,k_r]$, and for each $i \in \{1,2,...,q\}$, let S_i be the set of indices of the k_j's which correspond to the positions occupied by g_i:

$$S_i = \{j \in \{1,...,r\} : k_j = g_i\}.$$

From (6.8), it follows that if the T-expanded representation of the original system (with input $U(mT)$) is linked by the *matrix gain* $\Delta(r,S_i)$ to a duplicate of itself (with zero input) as described above, then the output of the duplicate system will be $\partial Y(mT)/\partial \ln|g_i|$.

A system in real time whose T-expanded representation is that proposed above to produce $\partial Y(mT)/\partial \ln|g_i|$ can easily be found. For $W \subseteq \{1,2,...,q\}$, let $\delta(nT_1,q,W)$ be the T_1q-periodic discrete-time function of n given by

$$\delta(nT_1,q,W) = \begin{cases} 1 \text{ , if } (1 + (n \bmod q)) \in W \\ 0 \text{ , otherwise} \end{cases}.$$

Assume $T/qT_1 = s_2$ as above; from (2.6), a SISO discrete-time periodic gain with input u_g and output y_g described by $y_g(nT_1) = \delta(nT_1,q,\{i\})\, u_g(nT_1)$ is seen to have the T-expanded representation $Y_g(mT) = \mathrm{diag}[\Delta(q,\{i\}),...,\Delta(q,\{i\})]\, U_g(mT)$, where $\Delta(q,\{i\})$ appears s_2 times along the diagonal. From $K = \mathrm{diag}[G,...,G]$, $\mathrm{diag}[\Delta(q,\{i\}),...,\Delta(q,\{i\})] = \Delta(r,S_i)$. Thus, if the signal entering the periodic gain $g(nT_1)$ in the original multirate discrete-time system is sampled at rate $1/T_1$, passed through the periodic gain $\delta(nT_1,q,\{i\})$, and summed with the signal entering the gain $g(nT_1)$ in a copy of the original system (with zero inputs,) the T-expanded representation of this composite system coincides with that described previously for determining $\partial Y(mT)/\partial \ln|g_i|$. Since $(\partial/\partial \ln|g_i|)$ can be distributed across the components of the vector $Y(mT)$, $\partial Y(mT)/\partial \ln|g_i|$ is the T-expanded version of $\partial y/\partial \ln|g_i|$. The meaning of $\partial y/\partial \ln|g_i|$ stems from the satisfaction of

Assumption 2.1(f); each output line y_j of the system is sampled and held at some rate $1/T_j$ and the corresponding output in the duplicate system, $y_{dj}(nT_j)$, gives $\partial y_j(nT_j)/\partial \ln|g_i|$.

The following procedure summarizes the variable component method for multirate discrete-time systems. This procedure is more general than the results discussed in this section, but it follows from linearity and the chain rule for differentiation.

Procedure 6.1:

a. Let the system have a given input u_0 and resulting output y_0. Verify that Assumption 2.1 is met, with the exception that periodic single-rate and multirate discrete-time subsystems may be present.

b. Suppose there is a periodic discrete-time gain $g(nT_1)$ in the system with period qT_1 and having values $\{g_1,\ldots,g_q\}$, as described above. $g(nT_1)$ may be either a purely discrete-time gain (such as a parameter in a computer program,) or the discretized version of a continuous-time gain with sample and hold devices at rate $1/T_1$ on its input and output. Let $W \subseteq \{1,2,\ldots,q\}$, and for all $i \in W$, let g_i be varied in unison and have value g. The parameters g_i, $i \notin W$, will be assumed independent of g.

c. Sample the signal entering $g(nT_1)$ on the block diagram of the system with rate $1/T_1$. Pass this sampled signal through the periodic gain $\delta(nT_1,q,W)$ and sum the output of this gain with the signal entering $g(nT_1)$ in a duplicate of the original system.

d. Repeat steps (b) and (c), using the <u>same</u> duplicate system, at each periodic discrete-time gain in the system that has parameters with value g if these parameters are to be varied in unison with the parameters in $g(nT_1)$. Both the rate and period of these gains may be different from location to location.

e. Simulate or implement the composite system resulting from the above steps. The input to the duplicate system should be set to zero, and the initial conditions on both the original and duplicate systems must be zero. The output of the duplicate system is then $\partial y_0/\partial \ln|g|$.

f. Add to the composite system an additional duplicate system connected to the original system for each additional parameter that is to be varied independently if all semirelative output sensitivity functions are to be obtained simultaneously.

♦♦♦

The variable component method described in Procedure 6.1 possesses properties which simplify the mechanics of applying Procedure 6.1. With reference to parts (b) and (c) of Procedure 6.1, note that $\delta(nT_1,q,W) = 1$ only at the times when $g(nT_1) = g$. This is seen to hold in general; at each discrete-time gain in the system, values are passed to the duplicate system only

at times when the parameter in question is "active" at that location. In particular, if $g(nT_1)$ is time invariant, $\delta(nT_1,q,W) \equiv 1$. Neither the T-expanded representation nor the value of T is needed to apply Procedure 6.1. As long as the resulting system satisfies Assumption 2.1 or its extensions, any number of signals in the system may be considered as outputs, and these outputs may be sampled at any rate. A change in the choice of outputs or their rates does not induce any change in the application of Procedure 6.1.

As with the variable component method for time-invariant parameters, Procedure 6.1 has the property that an additional duplicate system is required for each independently varied parameter. If each value of a periodic gain is to be varied independently, the number of duplicate systems and the associated computation may become excessive. Since the resulting systems will be periodic, it is in general insufficient to tune the response of the system to a fixed input, such as a step at time $t = 0$. Instead, it may be necessary to tune the response of the system to a given type of input, such as a step, applied at many individual times during the period of the system. A system with input $u(nT_1)$ exhibits T/T_1 step responses which are in general distinct.

6.3 Application of the Variable Component Method

The semirelative output sensitivity function may be used to tune the time-domain performance of a controlled system either intuitively or through the solution of an optimization problem. The variable component method described in Procedure 6.1 provides a means of obtaining this sensitivity function and has the advantages of being easy to apply and allowing the tuning of periodic gains at multiple rates. The computational burden of applying Procedure 6.1 may be substantial if the number of parameters to be varied independently is large. Conditions under which an iterative tuning scheme based on output sensitivity functions involving minimization of a cost function will converge or result in a stable system are in general unknown as of the present.

Controllers designed with the aid of the variable component method offer opportunities for taking full advantage of multirate sampling. Since Procedure 6.1 allows outputs to be sampled at any rate, the parameters of a single-rate controller for a continuous-time system can be adjusted according to a cost based on samples of the controlled system's output at a rate which is faster than the rate of the controller. Any accessible signal in the system can be considered as an output, enabling the parameter changes to reflect control effort. The controllers designed in Chapter 5 share the property that they are not directly implementable in a parallel form which capitalizes on the multirate nature of their inputs and outputs. An application of the variable component method might involve tuning multiple feedback loops containing single-rate controllers of a fixed, simple

structure (such as PID,) whose rates are commensurate with the time scales present in the system. Such controllers are readily implemented in hardware as independent parallel processing units or in software as concurrent tasks.

Example 6.1:

In Example 2.7, assume that it is desired to tune the system by varying g_1 and g_2 independently. To gain a more accurate picture of the behavior of the double integrator, define an additional output of the system as y_0, the output of the double integrator sampled with period $T_4 = 0.1$ sec. As y is a sampled version of y_0, it will henceforth be ignored. Application of Procedure 6.1 yields the composite system shown in Figure 6.3, where $y_1 = \partial y_0/\partial \ln|g_1| = (\partial y_0/\partial g_1)g_1$ and $y_2 = \partial y_0/\partial \ln|g_2|$.

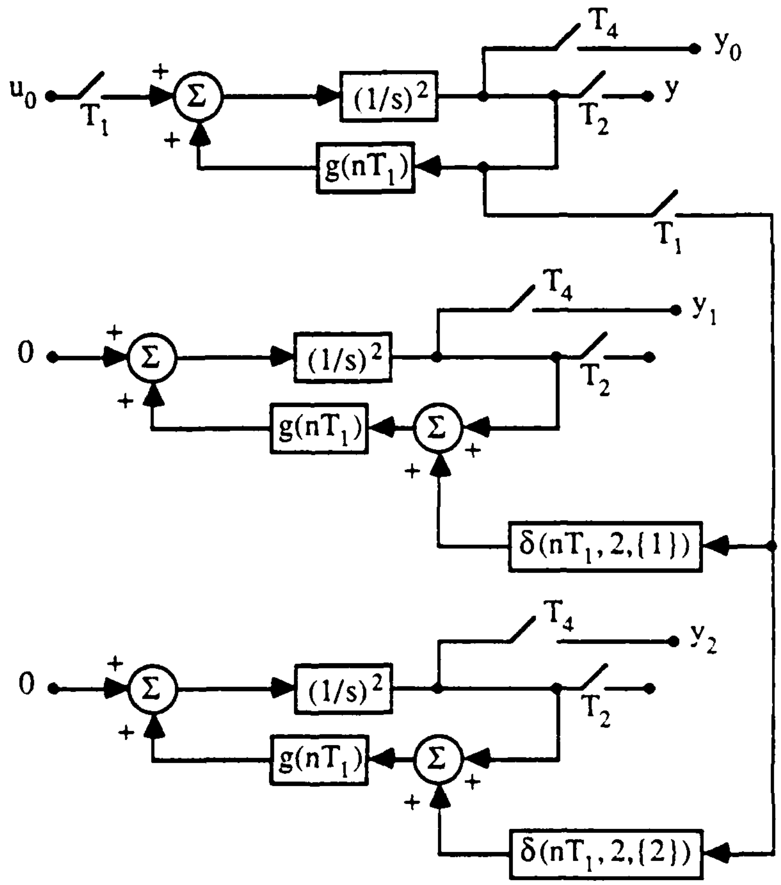

Figure 6.3. Composite system resulting from Procedure 6.1.

Denote the value of the output y_0 at time $t = mT_4$ with parameter values g_1 and g_2 by $y_0(mT_4,g_1,g_2)$. $y_1(mT_4,g_1,g_2)$ and $y_2(mT_4,g_1,g_2)$ are defined similarly. At this point, a number of options are available for tuning the system. With the intention of achieving reasonable settling time, overshoot, and rise time, consider choosing g_1 and g_2 to minimize the cost

$$J(g_1,g_2) = \sum_{m=0}^{200} (mT_4 [y_0(mT_4,g_1,g_2) - u_0(mT_4)])^2. \qquad (6.9)$$

$y_0(mT_4,g_1+\Delta g_1,g_2+\Delta g_2)$ can be approximated for small Δg_1 and Δg_2 by

$$y_0(mT_4,g_1,g_2) + [\partial y_0(mT_4,g_1,g_2)/\partial g_1]\, \Delta g_1 + [\partial y_0(mT_4,g_1,g_2)/\partial g_2]\, \Delta g_2,$$

which can be written as

$$y_0(mT_4,g_1,g_2) + (\Delta g_1/g_1)[y_1(mT_4,g_1,g_2)] + (\Delta g_2/g_2)[y_2(mT_4,g_1,g_2)]. \qquad (6.10)$$

Substituting (6.10) into (6.9) gives an approximation of $J(g_1+\Delta g_1,g_2+\Delta g_2)$ which is quadratic in Δg_1 and Δg_2 and can be minimized analytically in terms of $u_0(mT_4)$, $y_0(mT_4,g_1,g_2)$, $y_1(mT_4,g_1,g_2)$, and $y_2(mT_4,g_1,g_2)$ which result from simulation of the system in Figure 6.3 for fixed u_0, g_1, and g_2.

Suppose the initial parameter values are $g_1(0) = -1.5$ and $g_2(0) = 1$ (these parameter values stabilize the system) and the input u_0 is a 3 sec pulse at time $t = 0$ of unit height. The response of the composite system with $g_1(0)$ and $g_2(0)$ to u_0 is shown in Figure 6.4. Minimizing $J(g_1(0)+\Delta g_1(0),g_2(0)+\Delta g_2(0))$ gives as optimal values $\Delta g_1(0)/g_1(0) = 0.2633$ and $\Delta g_2(0)/g_2(0) = 0.4645$ and hence

$$g_1(1) = g_1(0) + \Delta g_1(0) = -1.895 \text{ and } g_2(1) = g_2(0) + \Delta g_2(0) = 1.465.$$

As shown in Figure 6.5, this choice of parameter change produces a weighted combination of y_1 and y_2 quite suited to reducing the magnitude of the oscillation in y_0. Figure 6.6 shows $y_0(mT_4,g_1(1),g_2(1))$, which is seen to have damping much improved over $y_0(mT_4,g_1(0),g_2(0))$.

After repeating the process of minimizing $J(g_1+\Delta g_1,g_2+\Delta g_2)$ a total of seven more times, the values of g_1 and g_2 for all practical purposes converge to $g_1^* = -2.714$ and $g_2^* = 2.008$. The response of the system to $u_0(nT_1)$ with these parameter values is shown in Figure 6.7. The state transition matrix for this system is unchanged from that of Example 2.7(i). Substituting g_1^* and g_2^* into this state transition matrix gives a matrix with eigenvalues -0.443 and 0.021. The final tuned system is thus asymptotically stable. Since $T/T_1 = 2$, this system exhibits two step responses. Figure 6.8 shows the response of the system with parameter values g_1^* and g_2^* to $u_0(nT_1-T_1)$. If the response in Figure 6.8 is deemed undesirable, one

might consider repeating the tuning process by minimizing $J(g_1+\Delta g_1, g_2+\Delta g_2)$ for both the input $u_0(nT_1)$ and the input $u_0(nT_1-T_1)$ at each stage, requiring two simulation runs for each iteration.

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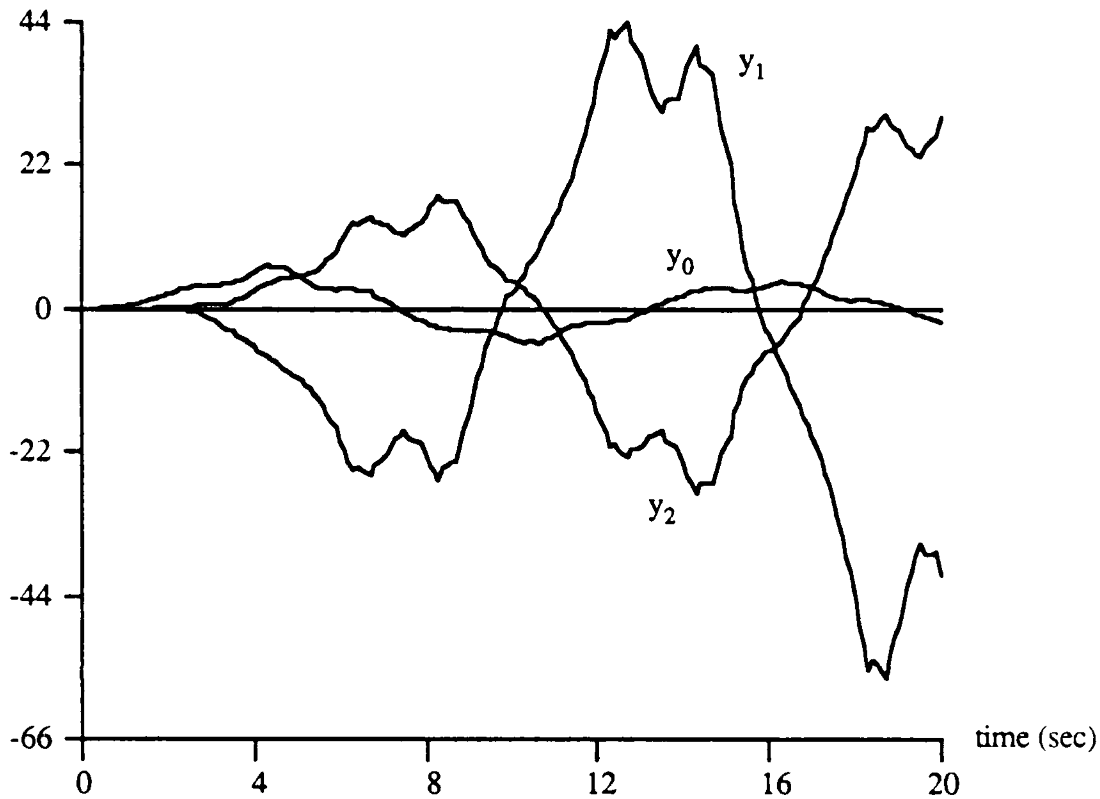

Figure 6.4. Response with $g_1(0) = -1.5$, $g_2(0) = 1.0$.

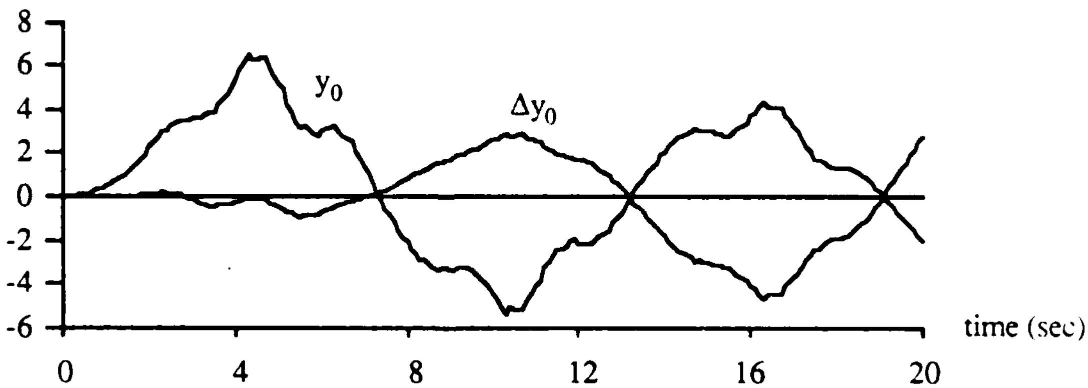

Figure 6.5. Response with $g_1(0) = -1.5$, $g_2(0) = 1.0$:
$\Delta y_0 = (\Delta g_1(0)/g_1(0))y_1 + (\Delta g_2(0)/g_2(0))y_2$.

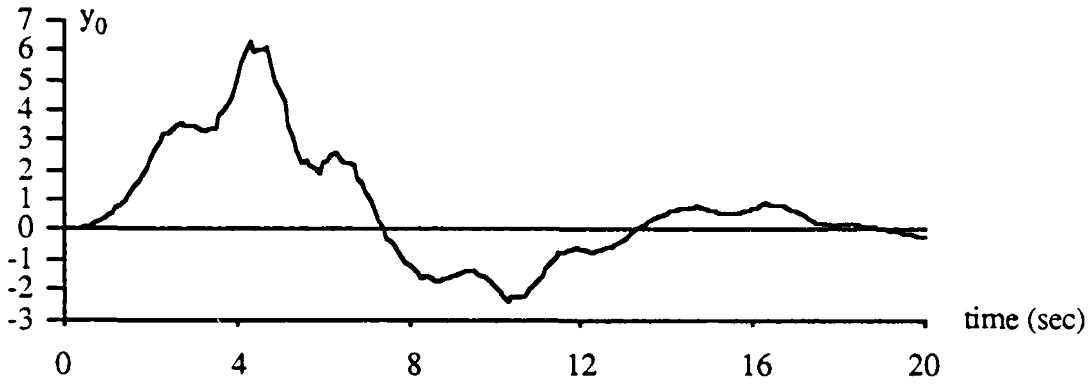

Figure 6.6. Response with $g_1(1) = -1.895$, $g_2(1) = 1.465$.

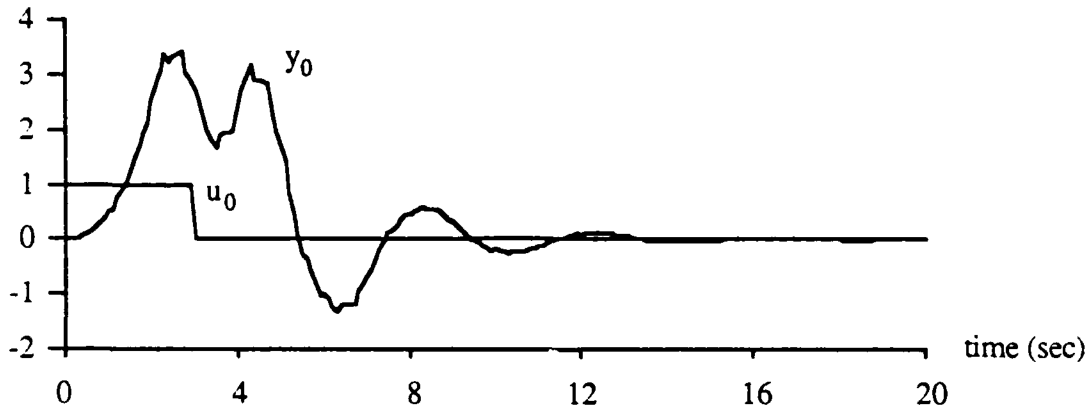

Figure 6.7. Response with $g_1^* = -2.714$, $g_2^* = 2.008$.

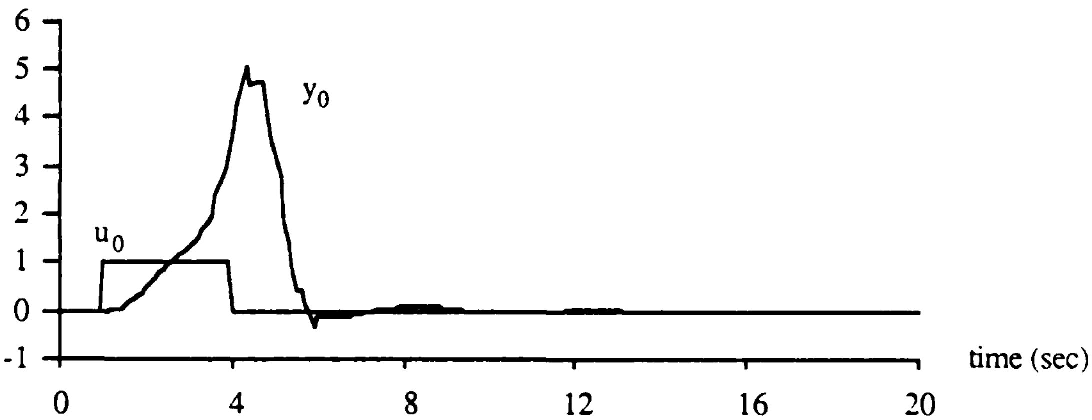

Figure 6.8. Response to $u_0(nT_1-T_1)$ with $g_1^* = -2.714$, $g_2^* = 2.008$.

CHAPTER 7

CONCLUSIONS

The results detailed in this work demonstrate the usefulness of the T-expanded representation of a multirate system. A detailed, systematic procedure was presented which enables the T-expanded representation of a multirate system to be calculated using any linear systems software package capable of matrix operations and single-rate discretization. A certain amount of information is clearly lost by expressing a multirate system in T-expanded form, most notably the interperiod state values. However, it was shown that the τ-varying representation inherits the following important qualities from its corresponding T-expanded representation: stability, controllability, reconstructibility, stabilizability, and detectability. Although this argument was carried out for the τ-varying representation presented here (or more generally, the M-varying representation,) the process of defining corresponding representations and proving the theorems in Chapter 3 and Chapter 4 could be carried out for almost any periodic representation of the class of multirate systems treated here. The multirate controller designs in Chapter 5 and the extension of the variable component method presented in Chapter 6 demonstrate that the T-expanded representation and the notation and concepts associated with it find practical and theoretical use in extending time-invariant results to multirate systems. Many extensions to the class of multirate systems for which T-expanded representations were defined are possible. For example, subsystems employing cyclic sampling (irregular sampling patterns which repeat periodically in time) could be treated by altering certain connection matrices to reflect the flow of information between the cyclic samplers and the other samplers in the system.

APPENDIX

SELECTED PROOFS

Proof of Theorem 2.1

Lemma A.1:

If the sampling periods $\{T_1,...,T_N\}$ are normalized with respect to T_k, $k \in \{1,...,N\}$, then GCD $\{p_{ki} : i = 1,...,N\} = 1$.

Proof: (by contradiction)

Without loss of generality, suppose that normalization is performed with respect to T_1. Observe that $r_{11} = q_{11} = 1$, since $T_1 = (1)T_1$. The case $N = 1$ is trivial, so consider only $N > 1$.

GCD $\{R_1/r_{11},...,R_1/r_{1N}\} = 1$ because $R_1 =$ LCM $\{r_{1i} : i = 1,...,N\}$. *Assume* that

$$\text{GCD } \{p_{1i} : i = 1,...,N\} = \text{GCD } \{R_1, R_1q_{12}/r_{12},..., R_1q_{1N}/r_{1N}\} = n \neq 1.$$

For this to be true, n must divide R_1. From GCD $\{R_1/r_{11},...,R_1/r_{1N}\} = 1$, there must be some $j \in \{2,3,...,N\}$ such that n is not a factor of R_1/r_{1j}, and (because n divides R_1) n is a factor of r_{1j}. For GCD $\{R_1q_{1i}/r_{1i} : i = 1,...,N\} = n$ to be true, n must be a factor of R_1q_{1j}/r_{1j}, so n is a factor of q_{1j}. This is a contradiction, because $n \neq 1$ and each r_{1i}/q_{1i} had common factors removed.

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Theorem 2.1:

If the sampling periods are normalized with respect to T_k (giving $T_i = p_{ki}\tau_k$, $i = 1,...,N$) and again with respect to T_j (giving $T_i = p_{ji}\tau_j$, $i = 1,...,N$,) then $\tau_k = \tau_j$ and $p_{ki} = p_{ji}$, $i = 1,...,N$.

Proof:

From $p_{ki}\tau_k = T_i = p_{ji}\tau_j$, $i = 1,...,N$, it follows that

$$\frac{\tau_j}{\tau_k} = \frac{p_{k1}}{p_{j1}} = \frac{p_{k2}}{p_{j2}} = \ldots = \frac{p_{kN}}{p_{jN}} .$$

Thus, τ_j/τ_k is a rational number. Let $\tau_j/\tau_k = n/m$, where $n,m \in \mathbb{N}$ and n and m have no common factor other than 1. From

$$n = \frac{mp_{k1}}{p_{j1}} = \ldots = \frac{mp_{kN}}{p_{jN}} ,$$

n is a common divisor of $\{p_{ki}m : i = 1,\ldots,N\}$. Since n and m have no common factors other than 1, n is a common divisor of $\{p_{ki} : i = 1,\ldots,N\}$. By Lemma A.1, $n \leq 1$. Thus, $n = 1$. Replacing n with m and k with j in the above argument yields $m = 1$. The statement of the theorem then follows.

♦♦♦

Proof of Theorem 2.2

Lemma A.2:

Given a set of sampling periods $\{T_1,\ldots,T_N\}$ which are expressed as $T_i = s_i\delta$, where $\delta \in \mathbb{R}$, $s_i \in \mathbb{N}$ for each i, and LCM $\{s_i : i = 1,\ldots,N\} = S$, the least common period of $\{T_1,\ldots,T_N\}$ is $S\delta$.

Proof: (by contradiction)

Assume LCP $\{T_1,\ldots,T_N\} = T_0 < S\delta$, and write $T_0 = \hat{S}\delta$, where $\hat{S} \in \mathbb{R}$ and $\hat{S} < S$. Then each sampler will sample an integral number of times during the time $\hat{S}\delta$;

$$\frac{\hat{S}\delta}{T_i} = \frac{\hat{S}\delta}{s_i\delta} = \frac{\hat{S}}{s_i} \in \mathbb{N} \text{ for } i = 1,\ldots,N.$$

This implies that $\hat{S} \geq$ LCM $\{s_i : i = 1,\ldots,N\} = S$, which is a contradiction.

♦♦♦

Theorem 2.2:

$T = \tau\,(\text{LCM } \{p_i : i = 1,\ldots,N\}) = P\tau$ is the shortest length of time over which a multirate system with sampling periods $\{T_1,\ldots,T_N\}$ is periodic.

Proof: A trivial application of Lemma A.2.

♦♦♦

Proof of Theorem 2.3

Theorem 2.3:

Let the LCP associated with $\{T_1,\ldots,T_k\}$ be $T(k)$ and that associated with $\{T_1,\ldots,T_k,T_{k+1}\}$ be $T(k+1)$. Then $T(k+1)/T(k) \in \mathbb{N}$.

Proof:

Normalization of $\{T_1,\ldots,T_k,T_{k+1}\}$ yields $T_i = p_i\tau$, $p_i \in \mathbb{N}$ for $i = 1,\ldots,k,k+1$. By Lemma A.2, $T(k) = \tau(\text{LCM }\{p_1,\ldots,p_k\})$ and $T(k+1) = \tau(\text{LCM }\{p_1,\ldots,p_{k+1}\})$. Since $\text{LCM }\{p_1,\ldots,p_{k+1}\} = \text{LCM }\{\text{LCM }\{p_1,\ldots,p_k\}, p_{k+1}\}$,

$$\frac{T(k+1)}{T(k)} = \frac{\tau(\text{LCM }\{p_1,\ldots,p_{k+1}\})}{\tau(\text{LCM }\{p_1,\ldots,p_k\})} = \frac{\text{LCM }\{\text{LCM }\{p_1,\ldots,p_k\}, p_{k+1}\}}{\text{LCM }\{p_1,\ldots,p_k\}} \in \mathbb{N}.$$

♦♦♦

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